

# THE ISAACS-NAVARRO CONJECTURE FOR COVERING GROUPS OF THE SYMMETRIC AND ALTERNATING GROUPS IN ODD CHARACTERISTIC

JEAN-BAPTISTE GRAMAIN

**ABSTRACT.** In this paper, we prove that a refinement of the Alperin-McKay Conjecture for  $p$ -blocks of finite groups, formulated by I. M. Isaacs and G. Navarro in 2002, holds for all covering groups of the symmetric and alternating groups, whenever  $p$  is an odd prime.

## 1. INTRODUCTION

I.M. Isaacs and G. Navarro have formulated in [5] some refinements of the McKay and Alperin-McKay Conjectures for arbitrary finite groups. Consider a finite group  $G$  and a prime  $p$ . Let  $B$  be a  $p$ -block of  $G$ , with defect group  $D$ , and let  $b$  be the Brauer correspondent of  $B$  in  $N_G(D)$ . Throughout this paper, we will use a  $p$ -valuation  $\nu$  on  $\mathbb{Z}$ , given by  $\nu(n) = a$  if  $n = p^a q$  with  $(p, q) = 1$ . The height  $\mathfrak{h}(\chi) \in \mathbb{Z}_{\geq 0}$  of an irreducible (complex) character  $\chi \in B$  is then defined by the equality  $\nu(\chi(1)) = \nu(|G|) - \nu(|D|) + \mathfrak{h}(\chi)$ . We denote by  $M(B)$  and  $M(b)$  the sets of characters of height 0 of  $B$  and  $b$  respectively. The Alperin-McKay Conjecture then asserts that  $|M(B)| = |M(b)|$  (while the McKay Conjecture states that  $|M(G)| = |M(N_G(P))|$ , where  $P \in \text{Syl}_p(G)$ , and  $M(G)$  and  $M(N_G(P))$  denote the sets of irreducible characters of  $p'$ -degree of  $G$  and  $N_G(P)$  respectively).

In [5], Isaacs and Navarro predicted that something stronger must happen, namely that this equality can be refined when considering the  $p'$ -parts of the character degrees. For any  $n \in \mathbb{N}$ , we write  $n = n_p n_{p'}$ , with  $n_p = p^{\nu(n)}$ . For any  $1 \leq k \leq p-1$ , we define subsets  $M_k(B)$  and  $M_k(b)$  of  $M(B)$  and  $M(b)$  respectively by letting  $M_k(B) = \{\chi \in M(B) ; \chi(1)_{p'} \equiv \pm k \pmod{p}\}$  and  $M_k(b) = \{\varphi \in M(b) ; \varphi(1)_{p'} \equiv \pm k \pmod{p}\}$ . We then have the following

**Conjecture 1.1.** [5, Conjecture B] *For  $1 \leq k \leq p-1$ , we have  $|M_{ck}(B)| = |M_k(b)|$ , where  $c = [G : N_G(D)]_{p'}$ .*

Note that Conjecture 1.1 obviously implies the Alperin-McKay Conjecture (by letting  $k$  run through  $\{1, \dots, p-1\}$ ), but also implies another refinement of the McKay Conjecture; if we let  $M_k(G) = \{\chi \in \text{Irr}(G) ; \chi(1) \equiv \pm k \pmod{p}\}$ , then, by considering all blocks of  $G$  with defect group  $P \in \text{Syl}_p(G)$ , we obtain  $|M_k(G)| = |M_k(N_G(P))|$ , since  $[G : N_G(P)] \equiv 1 \pmod{p}$  (see [5, Conjecture B]).

Isaacs and Navarro proved Conjecture 1.1 whenever  $D$  is cyclic, or  $G$  is  $p$ -solvable or sporadic. P. Fong proved it for symmetric groups  $S(n)$  in [2], and R. Nath for alternating groups  $A(n)$  in [8]. In this paper, we prove that Conjecture 1.1 holds in all the covering groups of the symmetric and alternating groups, provided  $p$  is odd (Theorem 5.1).

In Section 2, we present the covering groups  $S^+(n)$  and  $S^-(n)$  and their irreducible characters, first studied by I. Schur in [11], as well as their  $p$ -blocks. It turns out that the main work to be done is on so-called spin blocks. We also give various results on the degrees of spin characters, generalizing the methods used by Fong in [2]. Section 3 is devoted to proving Theorem 3.4 which reduces the problem to proving only that Conjecture 1.1 holds for the principal spin block of  $S^+(pw)$ . This reduction theorem is a refinement of [6, Theorem 2.2] that G. O. Michler and J. B. Olsson proved in order to establish that the Alperin-McKay Conjecture holds for covering groups. Finally, the case of the principal spin block of  $S^+(pw)$  is treated in Section 4.

## 2. COVERING GROUPS

In this section, we introduce the objects and preliminary results we will need about covering groups and their characters. Unless stated otherwise, the following results can be found in [6].

**2.1. Covering groups.** For any integer  $n \geq 1$ , I. Schur has defined (by generators and relations) two central extensions  $\hat{S}(n)$  and  $\tilde{S}(n)$  of the symmetric group  $S(n)$  (see [11], p. 164). We have  $\hat{S}(1) \cong \tilde{S}(1) \cong \mathbb{Z}/2\mathbb{Z}$ , and, for  $n \geq 2$ , there is a nonsplit exact sequence

$$1 \longrightarrow \langle z \rangle \longrightarrow \hat{S}(n) \xrightarrow{\pi} S(n) \longrightarrow 1,$$

where  $\langle z \rangle = Z(\hat{S}(n)) \cong \mathbb{Z}/2\mathbb{Z}$ .

Whenever  $n \geq 2$ , these two extensions are non-isomorphic, except when  $n = 6$ . However, their character tables are the same. Hence, for our purpose, it is sufficient to study one of them. Throughout this paper, we will write  $S^+(n)$  for  $\hat{S}(n)$ .

If  $H$  is a subgroup of  $S(n)$ , we let  $H^+ = \pi^{-1}(H)$  and  $H^- = \pi^{-1}(H \cap A(n))$ . In particular,  $H^-$  has index 1 or 2 in  $H^+$ , and  $H^+ = H^-$  if and only if  $H \subset A(n)$ . We define  $S^-(n) = A(n)^- = A(n)^+$ . Hence  $S^-(n)$  is a central extension of  $A(n)$  of degree 2.

The groups  $A(6)$  and  $A(7)$  also have one 6-fold cover each, which, together with the above groups, give all the covering groups of  $S(n)$  and  $A(n)$ .

**2.2. Characters, blocks and twisted central product.** From now on, we fix an odd prime  $p$ . For any  $H \leq S(n)$ , the irreducible complex characters of  $H^\varepsilon$  fall into two categories; those that have  $z$  in their kernel, and which can be identified with those of  $H$  (if  $\varepsilon = 1$ ) or those of  $H \cap A(n)$  (if  $\varepsilon = -1$ ), and those that don't have  $z$  in their kernel. These (faithful) characters are called *spin characters*. We denote by  $SI(H^\varepsilon)$  the set of spin characters of  $H^\varepsilon$ , and we let  $SI_0(H^\varepsilon) = SI(H^\varepsilon) \cap M(H^\varepsilon)$  (with the notation of Section 1).

If  $B$  is a  $p$ -block of  $H^\varepsilon$ , then, because  $p$  is odd, it is known that either  $B \cap SI(H^\varepsilon) = \emptyset$  or  $B \subset SI(H^\varepsilon)$ , in which case we say that  $B$  is a *spin block* of  $H^\varepsilon$ .

Any two  $\chi, \psi \in \text{Irr}(H^\varepsilon)$  are called *associate* if  $\chi \uparrow^{H^+} = \psi \uparrow^{H^+}$  (if  $\varepsilon = -1$ ) or if  $\chi \downarrow_{H^-} = \psi \downarrow_{H^-}$  (if  $\varepsilon = 1$ ). Then each irreducible character of  $H^\varepsilon$  has exactly 1 or 2 associate characters. If  $\chi$  is itself its only associate, we say that  $\chi$  is *self-associate* (written s.a.), we put  $\chi^a = \chi$ , and let  $\sigma(\chi) = 1$ . Otherwise,  $\chi$  has a unique associate  $\psi \neq \chi$ ; we say that  $\chi$  is *non-self-associate* (written n.s.a.), we put  $\chi^a = \psi$ , and we let  $\sigma(\chi) = -1$ .

If  $H^+ \neq H^-$ , then  $\chi \in \text{Irr}(H^+)$  and  $\varphi \in \text{Irr}(H^-)$  are said to *correspond* if  $\langle \chi, \varphi \uparrow^{H^+} \rangle_{H^+} \neq 0$ . In this case, Clifford's theory implies that  $\sigma(\chi) = -\sigma(\varphi)$ .

If  $H_1, H_2, \dots, H_k \leq S(n)$  act (non-trivially) on disjoint subsets of  $\{1, \dots, n\}$ , then one can define the *twisted central product*  $H^+ = H_1^+ \hat{\times} \dots \hat{\times} H_k^+ \leq S^+(n)$  (see [11] or [3]). Then  $|H^+| = \frac{1}{2^{k-1}} |H_1^+| \cdot |H_2^+| \dots |H_k^+| = 2 |H_1| \cdot |H_2| \dots |H_k|$ . Also, one obtains  $SI(H^+)$  from the  $SI(H_i^+)$ 's as follows:

**Proposition 2.1.** [11, §28] *There is a surjective map*

$$\hat{\otimes}: \begin{cases} SI(H_1^+) \times \dots \times SI(H_k^+) & \longrightarrow SI(H^+) \\ (\chi_1, \dots, \chi_k) & \longmapsto \chi_1 \hat{\otimes} \dots \hat{\otimes} \chi_k \end{cases}$$

which satisfies the following properties. Suppose  $\chi_i, \psi_i \in SI(H_i^+)$  for  $1 \leq i \leq k$ . Then

(i)  $\sigma(\chi_1 \hat{\otimes} \dots \hat{\otimes} \chi_k) = \sigma(\chi_1) \dots \sigma(\chi_k)$ , and  $(\chi_1 \hat{\otimes} \dots \hat{\otimes} \chi_k)(1) = 2^{\lfloor s/2 \rfloor} \chi_1(1) \dots \chi_k(1)$ , where  $s$  is the number of n.s.a. characters in  $\{\chi_1, \dots, \chi_k\}$  and  $\lfloor \cdot \rfloor$  denotes integral part.

(ii)  $\chi_1 \hat{\otimes} \dots \hat{\otimes} \chi_k$  and  $\psi_1 \hat{\otimes} \dots \hat{\otimes} \psi_k$  are associate if and only if  $\chi_i$  and  $\psi_i$  are associate for all  $i$ .

(iii)  $\chi_1 \hat{\otimes} \dots \hat{\otimes} \chi_k = \psi_1 \hat{\otimes} \dots \hat{\otimes} \psi_k$  if and only if  $\chi_i$  and  $\psi_i$  are associate for all  $i$  and  $|\sigma(\chi_1) \dots \sigma(\chi_k)| = 1$  or  $|\sigma(\chi_1) \dots \sigma(\chi_k)| = -1$  and  $|\{i \mid \chi_i \neq \psi_i\}|$  is even.

**2.3. Partitions and bar-partitions.** Just as the irreducible characters of  $S(n)$  are parametrized by the partitions of  $n$ , the spin characters of  $S^+(n)$  have a combinatorial description. We let  $P(n)$  be the set of all partitions of  $n$ , and  $P_0(n)$  be the subset of all partitions in distinct parts, also called *bar-partitions*. We write  $\lambda \vdash n$  for  $\lambda \in P(n)$ , and  $\lambda \succ n$  for  $\lambda \in P_0(n)$ . We also write, in both cases,  $|\lambda| = n$ .

It is well known that  $\text{Irr}(S(n)) = \{\chi_\lambda, \lambda \vdash n\}$ . For any  $\lambda \vdash n$ , we write  $h(\lambda)$  for the product of all hook-lengths in  $\lambda$ . We then have  $h(\lambda) = h_{\lambda, p} h_{\lambda, p'}$ , where  $h_{\lambda, p}$  (respectively  $h_{\lambda, p'}$ ) is the product of all hook-lengths divisible by  $p$  (prime to  $p$  respectively) in  $\lambda$ . The Hook-Length Formula then gives  $\chi_\lambda(1) = \frac{n!}{h(\lambda)}$ .

If we remove all the hooks of length divisible by  $p$  in  $\lambda$ , we obtain its  $p$ -core  $\lambda_{(p)}$ . The information on  $p$ -hooks is stored in the  $p$ -quotient  $\lambda^{(p)}$  of  $\lambda$ . If  $n = pw + r$ , with  $\lambda_{(p)} \vdash r$ , then  $\lambda^{(p)}$  is a  $p$ -tuple of partitions of  $w$ , i.e.  $\lambda^{(p)} = (\lambda^{(0)}, \dots, \lambda^{(p-1)})$  and  $|\lambda^{(0)}| + \dots + |\lambda^{(p-1)}| = w$ . The partition  $\lambda$  is uniquely determined by its  $p$ -core and  $p$ -quotient. Also, for any integer  $k$ , there exists a (canonical) bijection between the  $kp$ -hooks in  $\lambda$  and the  $k$ -hooks in  $\lambda^{(p)}$  (i.e. in the  $\lambda^{(i)}$ 's).

Finally, the Nakayama Conjecture states that  $\chi_\lambda, \chi_\mu \in \text{Irr}(S(n))$  belong to the same  $p$ -block if and only if  $\lambda$  and  $\mu$  have the same  $p$ -core.

We now present the analogue properties for bar-partitions and spin characters. For any bar-partition  $\lambda = (a_1, \dots, a_m)$  of  $n$ , with  $a_1 > \dots > a_m > 0$ , we let  $m(\lambda) = m$ , and define the *sign* of  $\lambda$  by  $\sigma(\lambda) = (-1)^{n-m(\lambda)}$ . We then have

**Theorem 2.2.** [11, §41] *For each sign  $\varepsilon \in \{1, -1\}$ , there is a (canonical) surjective map  $f^\varepsilon: SI(S^\varepsilon(n)) \longrightarrow P_0(n)$  such that:*

- (i)  $\sigma(\chi) = \varepsilon \sigma(f^\varepsilon(\chi))$  for all  $\chi \in SI(S^\varepsilon(n))$ .
- (ii) For any  $\chi, \psi \in SI(S^\varepsilon(n))$ , we have  $f^\varepsilon(\chi) = f^\varepsilon(\psi)$  if and only if  $\chi$  and  $\psi$  are associate.
- (iii) If  $\chi \in SI(S^+(n))$  and  $\varphi \in SI(S^-(n))$ , then  $f^+(\chi) = f^-(\varphi)$  if and only if  $\chi$  and  $\varphi$  correspond.

In particular, each  $\lambda \succ n$  labels one s.a. character  $\chi$  or two associate characters  $\chi$  and  $\chi^a$ . Throughout this paper, we will denote by  $\langle \lambda \rangle$  the set of spin characters labeled by  $\lambda$ , and write (abusively)  $\langle \lambda \rangle \in SI(S^\varepsilon(n))$ , and  $\langle \lambda \rangle(1)$  for the (common) degree of any spin character in  $\langle \lambda \rangle$ . We will also sometimes write  $\langle \lambda \rangle_+$  to emphasize that  $\langle \lambda \rangle \in SI(S^+(n))$  (and  $\langle \lambda \rangle_-$  if  $\langle \lambda \rangle \in SI(S^-(n))$ ).

For the following results on bars, cores and quotients, we refer to [9]. For any odd integer  $q$ , let  $e = (q-1)/2$ . We define a  $\bar{q}$ -quotient of weight  $w$  to be any tuple of partitions  $(\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(e)})$  such that  $\lambda^{(0)} \in P_0(w_0)$ ,  $\lambda^{(i)} \in P(w_i)$  for  $1 \leq i \leq e$ , and  $w_0 + w_1 + \dots + w_e = w$ . We define its *sign* by  $\sigma((\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(e)})) = (-1)^{w-w_0} \sigma(\lambda^{(0)})$ .

Now take any bar-partition  $\lambda = (a_1, \dots, a_m)$  of  $n$  as above. The *bars* in  $\lambda$  can be read in the *shifted Young diagram*  $S(\lambda)$  of  $\lambda$ . This is obtained from the usual Young diagram of  $\lambda$  by shifting the  $i$ -th row  $i-1$  positions to the right. The  $j$ -th node in the  $i$ -th row is called the  $(i, j)$ -node, and correspond to the *bar*  $B_{ij}$ . The *bar-lengths* in the  $i$ -th row are obtained by writing (from left to right in  $S(\lambda)$ ) the elements of the following set in decreasing order:  $\{1, 2, \dots, a_i\} \cup \{a_i + a_j \mid j > i\} \setminus \{a_i - a_j \mid j > i\}$ . The bars are of three types:

- Type 1. These are bars  $B_{ij}$  with  $i + j \geq m + 2$  (i.e. in the right part of  $S(\lambda)$ ). They are ordinary hooks in  $S(\lambda)$ , and their lengths are the elements of  $\{1, 2, \dots, a_i - 1\} \setminus \{a_i - a_j \mid j > i\}$ .
- Type 2. These are bars  $B_{ij}$  with  $i + j = m + 1$  (in particular, the corresponding nodes all belong to the same column of  $S(\lambda)$ ). Their length is precisely  $a_i$ , and the bar is all of the  $i$ -th row of  $S(\lambda)$ .
- Type 3. The lengths  $\{a_i + a_j \mid j > i\}$  correspond to bars  $B_{ij}$  with  $i + j \leq m$ . The bar consists of the  $i$ -th row together with the  $j$ -th row of  $S(\lambda)$ .

Bars of type 1 and 2 are called *unmixed*, while those of type 3 are called *mixed*. The unmixed bars in  $\lambda$  correspond exactly to the hooks in the partition  $\lambda^*$ , which admits as a  $\beta$ -set the set of parts of  $\lambda$ .

For any  $\lambda \succ n$ , we write  $\bar{h}(\lambda)$  for the product of all bar-lengths in  $\lambda$ . We then have  $\bar{h}(\lambda) = \bar{h}_{\lambda, p} \bar{h}_{\lambda, p'}$ , where  $\bar{h}_{\lambda, p}$  (respectively  $\bar{h}_{\lambda, p'}$ ) is the product of all bar-lengths divisible by  $p$  (prime to  $p$  respectively) in  $\lambda$ . We then have the following analogue of the Hook-Length Formula (proved by A. O. Morris, [7, Theorem 1])

$$\langle \lambda \rangle(1) = 2^{\lfloor (n-m(\lambda))/2 \rfloor} \frac{n!}{\bar{h}(\lambda)}.$$

If we remove all the bars of length divisible by  $p$  in  $\lambda$ , we obtain its  $\bar{p}$ -core  $\lambda_{(\bar{p})}$  (which is still a bar-partition), and its  $\bar{p}$ -quotient  $\lambda^{(\bar{p})}$ . If  $n = pw + r$ , with  $\lambda_{(\bar{p})} \succ r$ , then  $\lambda^{(\bar{p})}$  is a  $\bar{p}$ -quotient of weight  $w$  in the sense defined above. The bar-partition  $\lambda$  is uniquely determined by its  $\bar{p}$ -core and  $\bar{p}$ -quotient. Also, for any integer  $k$ , there exists a canonical bijection between the set of  $kp$ -bars in  $\lambda$  and the set of  $k$ -bars in  $\lambda^{(\bar{p})}$  (where a  $k$ -bar in  $\lambda^{(\bar{p})} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{((p-1)/2)})$  is a  $k$ -bar in  $\lambda^{(0)}$  or a  $k$ -hook in one of  $\lambda^{(1)}, \dots, \lambda^{((p-1)/2)}$ ).

The distribution of the spin characters of  $S^+(n)$  into spin blocks was first conjectured for  $p$  odd by Morris. It was first proved by J. F. Humphreys in [4], then differently by M. Cabanes, who also determined the structure of the defect groups of spin blocks (see [1]).

**Proposition 2.3.** *Let  $\chi, \psi \in SI(S^\varepsilon(n))$  and  $p$  be an odd prime. Then  $\chi$  is of  $p$ -defect 0 if and only if  $f^\varepsilon(\chi)$  is a  $\bar{p}$ -core. If  $f^\varepsilon(\chi)$  is not a  $\bar{p}$ -core, then  $\chi$  and  $\psi$  belong to the same  $p$ -block if and only if  $f^\varepsilon(\chi)_{(\bar{p})} = f^\varepsilon(\psi)_{(\bar{p})}$ .*

One can therefore define the  $\bar{p}$ -core of a spin block  $B$  and its *weight*  $w(B)$ , as well as its *sign*  $\delta(B) = \sigma(f^\varepsilon(\chi)_{(\bar{p})})$  (for any  $\chi \in B$ ). We then have

**Proposition 2.4.** [1] *If  $B$  is a spin block of  $S^\varepsilon(n)$  of weight  $w$ , then a defect group  $X$  of  $B$  is a Sylow  $p$ -subgroup of  $S^\varepsilon(pw)$ .*

**2.4. Removal of  $p$ -bars.** The following result is the bar-analogue of [2, Lemma 3.2]; it describes how the removal of  $p$ -bars affects the product of  $p'$ -bar-lengths.

**Proposition 2.5.** *Suppose  $\lambda \succ n$  has  $\bar{p}$ -core  $\lambda_{(\bar{p})}$ . Then*

$$\bar{h}_{\lambda, p'} \equiv \pm 2^{-a(\lambda)} \bar{h}_{\lambda_{(\bar{p})}, p'} = \pm 2^{-a(\lambda)} \bar{h}(\lambda_{(\bar{p})}) \pmod{p},$$

where  $a(\lambda)$  is the number of  $p$ -bars of type 3 to remove from  $\lambda$  to get  $\lambda_{(\bar{p})}$ .

*Proof.* Let  $B_{ij}$  be a  $p$ -bar in  $\lambda$  and  $\lambda - B_{ij}$  be the bar-partition obtained from  $\lambda$  by removing  $B_{ij}$ . We distinguish two cases, depending on whether  $B_{ij}$  is unmixed or mixed.

First suppose that  $B_{ij}$  is unmixed (i.e.  $i + j > m(\lambda)$ ). We start by examining the unmixed  $p'$ -bars in  $\lambda$  and  $\lambda - B_{ij}$ . These correspond, in the notation above, to the  $p'$ -hooks in  $\lambda^*$  and  $(\lambda - B_{ij})^*$  respectively (considering  $\lambda$  and  $\lambda - B_{ij}$  as  $\beta$ -sets). The set of parts of  $\lambda$  is  $X = \{a_1, \dots, a_m\}$ , and the set of non-zero parts of  $\lambda - B_{ij}$  is  $Y = \{a_1, \dots, a_{i-1}, a_i - p, a_{i+1}, \dots, a_m\}$  (or  $Y = \{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m\}$  if  $a_i = p$ ). The  $p'$ -hooks in  $\lambda^*$  (resp.  $(\lambda - B_{ij})^*$ ) therefore correspond to pairs  $(x, y)$ , with  $0 \leq x < y$ ,  $(y - x, p) = 1$ , and  $x \notin X$ ,  $y \in X$  (resp.  $x \notin Y$ ,  $y \in Y$ ).

If  $B_{ij}$  is of type 1 (i.e.  $i + j > m(\lambda) + 1$ ), then  $a_i - p > 0$ , so that  $|Y| = |X|$  and  $(\lambda - B_{ij})^* = \lambda^* - h$  for some  $p$ -hook  $h$  in  $\lambda$ . In this case, we are thus exactly in the same context as [2, Lemma 3.2], and we get  $h_{\lambda^*, p'} \equiv -h_{\lambda^* - h, p'} = -h_{(\lambda - B_{ij})^*, p'} \pmod{p}$ .

If, on the other hand,  $B_{ij}$  is of type 2 (i.e.  $i + j = m(\lambda) + 1$ ), then  $a_i - p = 0$ , and  $Y = X \setminus \{p\}$ . Note that, in this case,  $Y$  is not a  $\beta$ -set for a partition of  $|\lambda^*| - p$ , while  $Y \cup \{0\}$  is. The  $p'$ -hooks in  $(\lambda - B_{ij})^*$  correspond to either pairs  $(x, y)$  with  $y \neq a_i$ , which also correspond to  $p'$ -hooks in  $\lambda^*$ , or to pairs  $(p, y)$ , with  $y > p$  and  $y \in X$ . These new hooks have lengths  $(a_1 - p), \dots, (a_{i-1} - p)$ . Finally, some hooks have disappeared: those corresponding to pairs  $(x, p)$  with  $x < p$  and  $x \notin X$ . These have lengths  $(p - x)$ , for  $0 \leq x < p$  and  $x \notin \{a_{i+1}, \dots, a_m\}$ .

We now turn to the mixed  $p'$ -bars in  $\lambda$  and  $\lambda - B_{ij}$ . Suppose first that  $B_{ij}$  is of type 1. Then  $m(\lambda) = m(\lambda - B_{ij})$ . Suppose that

$$a_1 > \dots > a_{i-1} > a_{i+1} > \dots > a_k > a_i - p > a_{k+1} > \dots > a_m.$$

To prove the result, we can simply ignore the bar-lengths which are common to  $\lambda$  and  $\lambda - B_{ij}$ . The mixed bars which disappear when going from  $\lambda$  to  $\lambda - B_{ij}$  have lengths

$$(a_1 + a_i), (a_2 + a_i), \dots, (a_{i-1} + a_i)$$

$$\text{and } (a_i + a_{i+1}), (a_i + a_{i+2}), \dots, (a_i + a_m).$$

The mixed bars which appear have lengths

$$(a_1 + a_i - p), (a_2 + a_i - p), \dots, (a_{i-1} + a_i - p),$$

$$(a_{i+1} + a_i - p), \dots, (a_k + a_i - p) \text{ and } (a_i - p + a_{k+1}), \dots, (a_i - p + a_m).$$

If we then just consider the lengths not divisible by  $p$ , it is easy to see that we can pair the bars disappearing with those appearing. The pairs are of the form  $(b, b')$ , where  $b$  is a bar in  $\lambda$  and  $b'$  is a bar in  $\lambda - B_{ij}$ , and  $|b'| = |b| - p$ . We thus get, in this case,

$$\prod_{b \text{ mixed } p\text{-bar in } \lambda} |b| \equiv \prod_{b' \text{ mixed } p\text{-bar in } \lambda - B_{ij}} |b'| \pmod{p}.$$

Together with the equality obtained above for unmixed  $p'$ -bars, we obtain that, if  $B_{ij}$  is a  $p$ -bar of type 1 in  $\lambda$ , then  $\bar{h}_{\lambda, p'} \equiv \pm \bar{h}_{\lambda - B_{ij}, p'} \pmod{p}$ .

Now suppose that  $B_{ij}$  is of type 2, i.e.  $a_i = p$ . Then the mixed bars which disappear when going from  $\lambda$  to  $\lambda - B_{ij}$  have lengths  $(a_1 + p), (a_2 + p), \dots, (a_{i-1} + p)$  (call these  $A$ ) and  $(p + a_{i+1}), (p + a_{i+2}), \dots, (p + a_m)$  (call these  $B$ ), while no new mixed bar appears.

The bars disappearing in  $A$  are compensated for by the hooks appearing in  $(\lambda - B_{ij})^*$  in the study of unmixed bars above (since  $a_u + p \equiv a_u - p \pmod{p}$  for all  $1 \leq u \leq i-1$ , the  $p'$ -parts are congruent mod  $p$  when these are not divisible by  $p$ ).

On the other hand, since  $0 < a_m < \dots < a_{i+1} < a_i = p$ , all the bar-lengths in  $B$  are coprime to  $p$ , and their product is

$$(p + a_{i+1})(p + a_{i+2}) \dots (p + a_m) \equiv a_{i+1}a_{i+2} \dots a_m \pmod{p}.$$

Now the hooks disappearing in the above discussion of unmixed bars all have length prime to  $p$  except one (corresponding to  $x = 0$ ). The product of the lengths prime to  $p$  is thus

$$\prod_{0 < x < p, x \notin \{a_{i+1}, \dots, a_m\}} (p - x) \equiv (-1)^{p-1-m+i} \prod_{0 < x < p, x \notin \{a_{i+1}, \dots, a_m\}} x \pmod{p}.$$

Hence the product of the  $p'$ -hook-lengths disappearing and the  $p'$ -bar-lengths in  $B$  is congruent (mod  $p$ ) to

$$(-1)^{p-1-m+i} \prod_{0 < y < p} y = (-1)^{p-1-m+i} (p-1)! \equiv (-1)^{p-m+i} \pmod{p}$$

(by Wilson's Theorem). Finally, we obtain that, if  $B_{ij}$  is a  $p$ -bar of type 2 in  $\lambda$ , then  $\bar{h}_{\lambda, p'} \equiv (-1)^{p-m+i} \bar{h}_{\lambda - B_{ij}, p'} \pmod{p}$ .

We now suppose that  $B_{ij}$  is a  $p$ -bar of type 3 in  $\lambda$ , i.e.  $i < j$ ,  $a_i > a_j$  and  $a_i + a_j = p$ . The set of parts of  $\lambda$  is  $X = \{a_1, \dots, a_m\}$  and the set of parts of  $\lambda - B_{ij}$  is  $Y = \{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{j-1}, a_{j+1}, \dots, a_m\}$ . Ignoring as before the bars which are common to  $\lambda$  and  $\lambda - B_{ij}$ , we see that the unmixed bars which disappear from  $\lambda$  to  $\lambda - B_{ij}$  have lengths

$$(a_i - x) \quad (0 \leq x < a_i, x \notin \{a_m, \dots, a_{i+1}\})$$

$$\text{and } (a_j - x) \quad (0 \leq x < a_j, x \notin \{a_m, \dots, a_{j+1}\}),$$

while those appearing have lengths

$$(a_1 - a_i), \dots, (a_{i-1} - a_i), (a_1 - a_j), \dots, (a_{i-1} - a_j), \text{ and } (a_{i+1} - a_j), \dots, (a_{j-1} - a_j).$$

On the other hand, there is no mixed bar appearing, while the mixed bars disappearing have lengths

$$(a_1 + a_i), \dots, (a_{i-1} + a_i) \quad (\text{rows 1, ..., } i-1, \text{ column } i),$$

$$(a_i + a_{i+1}), \dots, (a_i + a_{j-1}), (a_i + a_j), \dots, (a_i + a_m) \quad (\text{row } i),$$

$$\begin{aligned}
& (a_1 + a_j), \dots, (a_{i-1} + a_j) \quad (\text{rows } 1, \dots, i-1, \text{ column } j), \\
& (a_{i+1} + a_j), \dots, (a_{j-1} + a_j) \quad (\text{rows } i+1, \dots, j-1, \text{ column } j), \\
& \text{and } (a_j + a_{j+1}), \dots, (a_j + a_m) \quad (\text{row } j).
\end{aligned}$$

Now, since  $a_i + a_j = p$ , we have, for any  $1 \leq k \leq m$ ,

$$a_k - a_i \equiv a_k + a_j \pmod{p} \quad \text{and} \quad a_k + a_i \equiv a_k - a_j \pmod{p}.$$

In particular,  $a_k - a_i$  (resp.  $a_k + a_i$ ) is coprime to  $p$  if and only if  $a_k + a_j$  (resp.  $a_k - a_j$ ) is coprime to  $p$ , and, in that case,

$$(a_k \pm a_i)_{p'} = a_k \pm a_i \equiv a_k \mp a_j \equiv (a_k \mp a_j)_{p'} \pmod{p}.$$

We thus have the following compensations between the appearing unmixed bars and the appearing mixed bars:

$$\begin{aligned}
& (a_1 - a_i), \dots, (a_{i-1} - a_i) \longleftrightarrow (a_1 + a_j), \dots, (a_{i-1} + a_j) \\
& (a_1 - a_j), \dots, (a_{i-1} - a_j) \longleftrightarrow (a_1 + a_i), \dots, (a_{i-1} + a_i) \\
& \text{and } (a_{i+1} - a_j), \dots, (a_{j-1} - a_j) \longleftrightarrow (a_i + a_{i+1}), \dots, (a_i + a_{j-1}).
\end{aligned}$$

This accounts for all the appearing (unmixed) bars, and we're left exactly with the following disappearing bar-lengths:

$$\begin{aligned}
& (a_i - x) \quad (0 \leq x < a_i, x \notin \{a_m, \dots, a_{i+1}\}) \quad \text{unmixed of type 1}, \\
& (a_j - x) \quad (0 \leq x < a_j, x \notin \{a_m, \dots, a_{j+1}\}) \quad \text{unmixed of type 2}, \\
& (a_{i+1} + a_j), \dots, (a_{j-1} + a_j), (a_j + a_{j+1}), \dots, (a_j + a_m) \quad \text{mixed of type 1}, \\
& (a_i + a_{j+1}), \dots, (a_i + a_m) \quad \text{mixed of type 2},
\end{aligned}$$

and  $(a_i + a_j) = p$  which can thus be ignored.

Now, for any  $i+1 \leq k \leq m$ ,  $a_j + a_k = p - a_i + a_k \equiv -(a_i - a_k) \pmod{p}$ , and, for  $j+1 \leq k \leq m$ ,  $a_i + a_k \equiv -(a_j - a_k) \pmod{p}$ . Hence, taking the product, we obtain (modulo  $p$ ):

$$\prod_{0 \leq x < a_i, x \neq a_j} (a_i - x) = \frac{a_i!}{a_i - a_j} \quad (\text{type 1}) \quad \text{and} \quad \prod_{0 \leq x < a_j} (a_j - x) = a_j! \quad (\text{type 2}).$$

Now  $a_j! = 1 \cdot 2 \dots a_j = (-1)^{a_j} (-1) \dots (-a_j) \equiv (-1)^{a_j} (p-1) \dots (p-a_j) \pmod{p}$ , so that  $a_j! \equiv (-1)^{a_j} (p-1) \dots (a_i+1)a_i \pmod{p}$ . We thus have, disappearing,

$$\pm \frac{a_i a_i! (a_i+1) \dots (p-1)}{a_i - a_j} \equiv \pm \frac{a_i}{a_i - a_j} (p-1)! \equiv \mp \frac{a_i}{a_i - a_j} \pmod{p}$$

(this last equality being true by Wilson's Theorem).

Finally,  $a_i - a_j = a_i - (p - a_i) \equiv -2a_i \pmod{p}$ , yielding a total of  $\pm 2^{-1} \pmod{p}$  disappearing (since,  $p$  being odd, 2 is invertible  $\pmod{p}$ , and  $a_i < p$  so that we can simplify by  $a_i$ ). We thus get that, if  $B_{ij}$  is a  $p$ -bar of type 3 in  $\lambda$ , then  $\bar{h}_{\lambda, p'} \equiv \pm \frac{1}{2} \bar{h}_{\lambda - B_{ij}, p'} \pmod{p}$ .

Iterating the above results on all the  $p$ -bars to remove from  $\lambda$  to get to its  $\bar{p}$ -core  $\lambda_{(\bar{p})}$ , we finally obtain the desired equality, writing  $a(\lambda)$  for the number of  $p$ -bars of type 3 to remove:

$$\bar{h}_{\lambda, p'} \equiv \pm 2^{-a(\lambda)} \bar{h}_{\lambda_{(\bar{p})}, p'} = \pm 2^{-a(\lambda)} \bar{h}(\lambda_{(\bar{p})}) \pmod{p}$$

(since all the bars in  $\lambda_{(\bar{p})}$  have length coprime to  $p$ ).

□

**2.5.  $\bar{p}$ -core tower,  $\bar{p}$ -quotient tower and characters of  $p'$ -degree.** In this section, we want to obtain an expression for the (value modulo  $p$  of the)  $p'$ -part of the degree of a spin character. We start by describing the  $\bar{p}$ -core tower of a bar-partition, introduced by Olsson in [9].

Take any  $\lambda \succ n$ . the  $\bar{p}$ -core tower of  $\lambda$  has rows  $R_0^\lambda, R_1^\lambda, R_2^\lambda, \dots$ , where the  $i$ -th row  $R_i^\lambda$  contains one  $\bar{p}$ -core and  $(p^i - 1)/2$   $p$ -cores (in particular, one can consider  $R_0^\lambda$  as a  $\bar{p}^i$ -quotient). We have  $R_0^\lambda = \{\lambda_{(\bar{p})}\}$  (the  $\bar{p}$ -core of  $\lambda$ ). If the  $\bar{p}$ -quotient of  $\lambda$  is  $\lambda^{(\bar{p})} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(e)})$  (where  $e = (p-1)/2$ ), then  $R_1^\lambda = \{\lambda_{(\bar{p})}^{(0)}, \lambda_{(p)}^{(1)}, \dots, \lambda_{(p)}^{(e)}\}$ . Writing  $\lambda^{(0)(\bar{p})} = (\lambda^{(0,0)}, \lambda^{(0,1)}, \dots, \lambda^{(0,e)})$  the  $\bar{p}$ -quotient of  $\lambda^{(0)}$  and  $\lambda^{(i)(p)} = (\lambda^{(i,1)}, \lambda^{(i,2)}, \dots, \lambda^{(i,p)})$  the  $p$ -quotient of  $\lambda^{(i)}$  ( $1 \leq i \leq e$ ), and taking cores, we let

$$R_2^\lambda = \{\lambda_{(\bar{p})}^{(0,0)}, \lambda_{(p)}^{(0,1)}, \dots, \lambda_{(p)}^{(0,e)}, \lambda_{(p)}^{(1,1)}, \dots, \lambda_{(p)}^{(1,p)}, \lambda_{(p)}^{(2,1)}, \dots, \lambda_{(p)}^{(e,p)}\}.$$

Continuing in this way, we obtain the  $\bar{p}$ -core tower of  $\lambda$ . We define the  $\bar{p}$ -quotient tower of  $\lambda$  in a similar fashion: it has rows  $Q_0^\lambda, Q_1^\lambda, Q_2^\lambda, \dots$ , where the  $i$ -th row  $Q_i^\lambda$  contains one  $\bar{p}$ -quotient and  $(p^i - 1)/2$   $p$ -quotients (in particular,  $Q_i^\lambda$  can be seen as a  $\bar{p}^{i+1}$ -quotient). With the above notation, we have  $Q_0^\lambda = \{\lambda^{(\bar{p})}\}$ ,  $Q_1^\lambda = \{\lambda^{(0)(\bar{p})}, \lambda^{(1)(p)}, \dots, \lambda^{(e)(p)}\}$  and

$$Q_2^\lambda = \{\lambda^{(0,0)(\bar{p})}, \lambda^{(0,1)(p)}, \dots, \lambda^{(0,e)(p)}, \lambda^{(1,1)(p)}, \dots, \lambda^{(1,p)(p)}, \lambda^{(2,1)(p)}, \dots, \lambda^{(e,p)(p)}\}.$$

The following result will be useful later.

**Lemma 2.6.** *If  $\lambda \succ n$  has  $\bar{p}$ -core tower  $(R_0^\lambda, R_1^\lambda, \dots, R_m^\lambda)$ , then  $\sigma(\lambda) = \prod_{i=0}^m \sigma(R_i^\lambda)$ .*

*Proof.* We have  $\sigma(\lambda) = \sigma(\lambda_{(\bar{p})})\sigma(\lambda^{(\bar{p})})$ , and  $\sigma(\lambda_{(\bar{p})}) = \sigma(R_0^\lambda)$ .

Also,  $\sigma(\lambda^{(\bar{p})}) = \sigma(\lambda^{(0)})(-1)^{\sum_{i \geq 1} |\lambda^{(i)}|}$ , and

$$\sigma(\lambda^{(0)}) = \sigma(\lambda_{(\bar{p})}^{(0)})\sigma(\lambda^{(0)(\bar{p})}) = \sigma(Q_0^\lambda) = \sigma(\lambda_{(\bar{p})}^{(0)})\sigma(\lambda^{(0,0)})(-1)^{\sum_{j \geq 1} |\lambda^{(0,j)}|}.$$

Now  $\sigma(R_1^\lambda) = \sigma(\lambda^{(0)(\bar{p})})(-1)^{\sum_{i \geq 1} |\lambda_{(\bar{p})}^{(i)}|}$  and  $\sigma(Q_1^\lambda) = \sigma(\lambda^{(0,0)})(-1)^{\sum_{i \geq 0, j \geq 1} |\lambda^{(i,j)}|}$ , so that

$$\begin{aligned} \sigma(R_1^\lambda)\sigma(Q_1^\lambda) &= \sigma(\lambda^{(0)(\bar{p})})\sigma(\lambda^{(0,0)})(-1)^{\sum_{i \geq 1} |\lambda_{(\bar{p})}^{(i)}| + \sum_{i \geq 0, j \geq 1} |\lambda^{(i,j)}|} \\ &= \sigma(\lambda^{(0)})(-1)^{\sum_{i \geq 1} |\lambda_{(\bar{p})}^{(i)}| + \sum_{i,j \geq 1} |\lambda^{(i,j)}|}. \end{aligned}$$

However, for each  $i \geq 1$ , we have  $|\lambda_{(\bar{p})}^{(i)}| + \sum_{j \geq 1} |\lambda^{(i,j)}| \equiv |\lambda_{(\bar{p})}^{(i)}| + p \sum_{j \geq 1} |\lambda^{(i,j)}| \pmod{2}$  (since  $p$  is odd), and  $|\lambda_{(\bar{p})}^{(i)}| + p \sum_{j \geq 1} |\lambda^{(i,j)}| = |\lambda^{(i)}|$ . We therefore get

$$\sigma(R_1^\lambda)\sigma(Q_1^\lambda)\sigma(\lambda^{(0)})(-1)^{\sum_{i \geq 1} |\lambda^{(i)}|} = \sigma(\lambda^{(\bar{p})}) = \sigma(Q_0^\lambda).$$

Finally, we have  $\sigma(\lambda) = \sigma(R_0^\lambda)\sigma(Q_0^\lambda)$ , and  $\sigma(Q_0^\lambda) = \sigma(R_1^\lambda)\sigma(Q_1^\lambda)$ , whence  $\sigma(\lambda) = \sigma(R_0^\lambda)\sigma(R_1^\lambda)\sigma(Q_1^\lambda)$ . Iterating this process, we deduce the result.  $\square$

Now, writing  $\beta_i(\lambda)$  for the sum of the cardinalities of the partitions in  $R_i^\lambda$ , one shows easily that  $|\lambda| = \sum_{i \geq 0} \beta_i(\lambda)p^i$  (see [9]). Also, one gets the following bar-analogue of [2, Proposition 1.1]:

**Proposition 2.7.** [9, Proposition (3.1)] *In the above notation,*

$$\nu_p(\bar{h}(\lambda)) = \frac{n - \sum_{i \geq 0} \beta_i(\lambda)}{p-1}.$$

In particular,  $\langle \lambda \rangle$  has  $p'$ -degree if and only if  $\sum_{i \geq 0} \beta_i(\lambda)p^i$  is the  $p$ -adic decomposition of  $n$ .

Let  $n = \sum_{i=0}^k t_i p^i$  be the  $p$ -adic decomposition of  $n$ . For each  $0 \leq i \leq k$ , let  $e_i = (p^i - 1)/2$ , and write  $R_i^\lambda = \{\mu_i^{(0)}, \mu_i^{(1)}, \dots, \mu_i^{(e_i)}\}$  and  $Q_i^\lambda = \{\lambda_i^{(0)}, \lambda_i^{(1)}, \dots, \lambda_i^{(e_{i+1})}\}$ . Note that  $Q_k^\lambda = \{\emptyset, \dots, \emptyset\}$ .

We let  $\bar{h}(R_i^\lambda) = \bar{h}(\mu_i^{(0)}) \prod_{j=1}^{e_i} h(\mu_i^{(j)})$ , and  $\bar{h}(Q_i^\lambda) = \bar{h}(\lambda_i^{(0)}) \prod_{j=1}^{e_{i+1}} h(\lambda_i^{(j)})$ , and we let  $m_i^{(0)} = m(\mu_i^{(0)})$  and  $\beta_i = \beta_i(\lambda)$ .

**Proposition 2.8.** *With the above notation, we have, for any  $\lambda \succ n$ ,*

$$\frac{|S^+(n)|_{p'}}{\langle \lambda \rangle(1)_{p'}} \equiv \pm \frac{2}{2^{\lfloor \frac{S}{2} \rfloor}} \prod_{i=0}^k \frac{1}{2^{\lfloor (\beta_i - m_i^{(0)})/2 \rfloor}} \bar{h}(R_i^\lambda) \pmod{p},$$

where  $S = |\{0 \leq i \leq k; \beta_i - m_i^{(0)} \text{ odd}\}|$ .

*Proof.* We have

$$\frac{|S^+(n)|_{p'}}{\langle \lambda \rangle(1)_{p'}} = \frac{2}{2^{\lfloor (n - m(\lambda))/2 \rfloor}} \bar{h}(\lambda)_{p'}.$$

Now  $\bar{h}(\lambda)_{p'} = \bar{h}_{\lambda, p'}(\bar{h}_{\lambda, p})_{p'} = \bar{h}_{\lambda, p'}(\bar{h}(Q_0^\lambda)_{p'})$  (since there is a bijection between the set of bars divisible by  $p$  in  $\lambda$  and the set of bars in the quotient  $Q_0^\lambda$ ).

By Proposition 2.5, we have  $\bar{h}_{\lambda, p'} \equiv \pm 2^{-a(\lambda)} \bar{h}(\lambda_{(p)}) \equiv \pm 2^{-a(\lambda)} \bar{h}(R_0^\lambda) \pmod{p}$ . Also,

$$\begin{aligned} \bar{h}(Q_0^\lambda)_{p'} &= \bar{h}_{Q_0^\lambda, p'}(\bar{h}_{Q_0^\lambda, p})_{p'} \\ &= \bar{h}_{\lambda_0^{(0)}, p'} h_{\lambda_0^{(1)}, p'} \dots h_{\lambda_0^{(e_1)}, p'} \bar{h}(Q_1^\lambda)_{p'} \\ &\equiv \pm 2^{-a(\lambda_0^{(0)})} \bar{h}_{\mu_1^{(0)}, p'} h_{\mu_1^{(1)}, p'} \dots h_{\mu_1^{(e_1)}, p'} \bar{h}(Q_1^\lambda)_{p'} \pmod{p}, \end{aligned}$$

this last equality holding by 2.5 (applied to  $\lambda_0^{(0)}$ ) and by [2, Lemma 3.2] (applied to  $\lambda_0^{(1)}, \dots, \lambda_0^{(e_1)}$ ). We thus get  $\bar{h}(Q_0^\lambda)_{p'} \equiv \pm 2^{-a(\lambda_0^{(0)})} \bar{h}(R_1^\lambda) \bar{h}(Q_1^\lambda)_{p'} \pmod{p}$ , and

$$\bar{h}(\lambda)_{p'} \equiv \pm 2^{-a(\lambda) - a(\lambda_0^{(0)})} \bar{h}(R_0^\lambda) \bar{h}(R_1^\lambda) \bar{h}(Q_1^\lambda)_{p'} \pmod{p}.$$

Iterating this, until we get to  $Q_k^\lambda = \{\emptyset, \dots, \emptyset\}$ , we obtain

$$\bar{h}(\lambda)_{p'} \equiv \pm 2^{-\lfloor a(\lambda) + a(\lambda_0^{(0)}) + a(\lambda_1^{(0)}) + \dots + a(\lambda_{k-1}^{(0)}) \rfloor} \prod_{i=0}^k \bar{h}(R_i^\lambda) \pmod{p}.$$

On the other hand, repeated use of [9, Corollary 2.6] yields

$$\begin{aligned} m(\lambda) &= m_0^{(0)} + m(\lambda_0^{(0)}) + 2a(\lambda) \\ &= m_0^{(0)} + m_1^{(0)} + m(\lambda_1^{(0)}) + 2a(\lambda) + 2a(\lambda_0^{(0)}) \\ &= (\dots) \\ &= m_0^{(0)} + m_1^{(0)} + \dots + m_k^{(0)} + 2(a(\lambda) + a(\lambda_0^{(0)}) + \dots + a(\lambda_{k-1}^{(0)})), \end{aligned}$$

so that

$$\begin{aligned} \lfloor \frac{(n - m(\lambda))}{2} \rfloor &= \lfloor \frac{(n - m_0^{(0)} - \dots - m_k^{(0)})}{2} - (a(\lambda) + a(\lambda_0^{(0)}) + \dots + a(\lambda_{k-1}^{(0)})) \rfloor \\ &= \lfloor \frac{(n - (m_1^{(0)} + \dots + m_k^{(0)}))}{2} - (a(\lambda) + a(\lambda_0^{(0)}) + \dots + a(\lambda_{k-1}^{(0)})) \rfloor. \end{aligned}$$

Together with the expression we obtained for  $\bar{h}(\lambda)_{p'}$ , this gives

$$\frac{|S^+(n)|_{p'}}{\langle \lambda \rangle(1)_{p'}} \equiv \pm \frac{2}{2^{\lfloor (n-m_0^{(0)}-\dots-m_k^{(0)})/2 \rfloor}} \prod_{i=0}^k \bar{h}(R_i^\lambda) \pmod{p}.$$

Now recall that  $n = \sum_{i=0}^k \beta_i p^i$ . Also, for any  $1 \leq i \leq k$ , we have

$$\begin{aligned} \lfloor \frac{\beta_i p^i - m_i^{(0)}}{2} \rfloor &= \lfloor \frac{\beta_i(p^i-1)}{2} + \frac{\beta_i - m_i^{(0)}}{2} \rfloor \\ &= \lfloor \frac{(p-1)\beta_i(1+p+\dots+p^{i-1})}{2} + \frac{\beta_i - m_i^{(0)}}{2} \rfloor \\ &= \frac{(p-1)}{2} \beta_i(1+p+\dots+p^{i-1}) + \lfloor \frac{\beta_i - m_i^{(0)}}{2} \rfloor, \end{aligned}$$

and

$$2^{\frac{(p-1)}{2} \beta_i(1+p+\dots+p^{i-1})} = (2^{\frac{(p-1)}{2}})^{\beta_i(1+\dots+p^{i-1})} \equiv (-1)^{\beta_i(1+\dots+p^{i-1})} \equiv \pm 1 \pmod{p}.$$

Hence

$$\begin{aligned} 2^{\lfloor \frac{n-(m_0^{(0)}+\dots+m_k^{(0)})}{2} \rfloor} &= 2^{\lfloor \sum_{i=0}^k \frac{\beta_i p^i - m_i^{(0)}}{2} \rfloor} \\ &= 2^{\lfloor \sum_{i=1}^k \frac{p-1}{2} \beta_i(1+\dots+p^{i-1}) + \sum_{i=0}^k \frac{\beta_i - m_i^{(0)}}{2} \rfloor} \\ &= 2^{\sum_{i=1}^k \frac{p-1}{2} \beta_i(1+\dots+p^{i-1}) + \lfloor \sum_{i=0}^k \frac{\beta_i - m_i^{(0)}}{2} \rfloor} \\ &\equiv \pm 2^{\lfloor \sum_{i=0}^k \frac{\beta_i - m_i^{(0)}}{2} \rfloor} \pmod{p}. \end{aligned}$$

Now

$$\begin{aligned} \lfloor \sum_{i=0}^k \frac{\beta_i - m_i^{(0)}}{2} \rfloor &= \lfloor \sum_{i=0, \beta_i - m_i^{(0)} \text{ even}}^k \frac{\beta_i - m_i^{(0)}}{2} + \sum_{i=0, \beta_i - m_i^{(0)} \text{ odd}}^k \frac{\beta_i - m_i^{(0)}}{2} \rfloor \\ &= \sum_{i=0, \beta_i - m_i^{(0)} \text{ even}}^k \lfloor \frac{\beta_i - m_i^{(0)}}{2} \rfloor + \sum_{i=0, \beta_i - m_i^{(0)} \text{ odd}}^k \lfloor \frac{\beta_i - m_i^{(0)}}{2} \rfloor \end{aligned}$$

and we have  $\lfloor \sum_{i=0, \beta_i - m_i^{(0)} \text{ odd}}^k \frac{\beta_i - m_i^{(0)}}{2} \rfloor = \lfloor \frac{S}{2} \rfloor + \sum_{i=0, \beta_i - m_i^{(0)} \text{ odd}}^k \lfloor \frac{\beta_i - m_i^{(0)}}{2} \rfloor$ , where  $S = |\{0 \leq i \leq k; \beta_i - m_i^{(0)} \text{ odd}\}|$ . We finally obtain

$$\frac{|S^+(n)|_{p'}}{\langle \lambda \rangle(1)_{p'}} \equiv \pm \frac{2}{2^{\lfloor \frac{S}{2} \rfloor}} \prod_{i=0}^k \frac{1}{2^{\lfloor (\beta_i - m_i^{(0)})/2 \rfloor}} \bar{h}(R_i^\lambda) \pmod{p}.$$

□

### 3. REDUCTION THEOREM

In this section, we show that, in order to prove Conjecture 1.1 for any spin block  $B$  of  $S^\varepsilon(n)$  of positive weight  $w$ , it is enough to prove it for the principal spin block of  $S^+(pw)$  (i.e. that with empty  $\bar{p}$ -core). Our main tool to navigate between  $S^+(n)$  and  $S^-(n)$  is the strong duality that exists between their spin blocks.

Let  $H \leq S(n)$ . A block of  $H^\varepsilon$  is called *proper* if it contains both a s.a. character and a n.s.a. character. By [10, 2.1], any spin block of  $S^\varepsilon(n)$  of positive weight is proper. Now, if  $B$  is a proper block of  $H^\varepsilon$ , then  $H^\varepsilon \neq H^{-\varepsilon}$ , and there exists a unique block  $B^*$  of  $H^{-\varepsilon}$  covering  $B$  (if  $\varepsilon = -1$ ) or covered by  $B$  (if  $\varepsilon = 1$ ), and  $B^*$  is also proper. We say that  $B$  and  $B^*$  are *(dual) corresponding blocks*. Finally, if  $B$  is proper, then it follows that  $B$  consists of s.a. characters and pairs of n.s.a. characters. In particular, we can still write (abusively)  $\langle \lambda \rangle \in B$  or  $\langle \lambda \rangle \in M(B)$ . Also, for any sign  $\varepsilon$ , if  $\langle \lambda \rangle_\varepsilon \in SI(S^\varepsilon(n))$ , then we call  $\langle \lambda \rangle_{-\varepsilon} \in SI(S^{-\varepsilon}(n))$  the *dual correspondent* of  $\langle \lambda \rangle_\varepsilon$ .

**3.1. Preliminaries: the case  $\varepsilon = 1$ .** Let  $B$  be a spin block of  $S^+(n)$  of weight  $w = w(B) > 0$  and sign  $\delta = \delta(B)$ , and let  $B_0$  be the principal spin block of  $S^\delta(pw)$ . Let  $r = n - wp$ . Let  $\mu$  be the  $\bar{p}$ -core of  $B$ , so that  $\sigma(\mu) = \delta$ . The characters in  $B$  are indexed by the  $\bar{p}$ -quotients of weight  $w$ . For any bar-partition  $\lambda$  with  $\bar{p}$ -core  $\mu$ , we denote the  $\bar{p}$ -quotient of  $\lambda$  by  $\lambda^{(\bar{p})}$  (so that  $\sigma(\lambda) = \delta\sigma(\lambda^{(\bar{p})})$ ), and we let  $\tilde{\lambda}$  be the bar-partition of  $wp$  with empty  $\bar{p}$ -core, and  $\bar{p}$ -quotient  $\lambda^{(\bar{p})}$ .

**Lemma 3.1.** *If  $\delta = 1$ , then, with the above notation,  $\lambda \mapsto \tilde{\lambda}$  induces a sign-preserving bijection  $\mathcal{I}$  between  $B$  and  $B_0$  which is also height-preserving. Furthermore,*

$$\langle \lambda \rangle(1)_{p'} \equiv \pm \frac{(n!)_{p'}}{((wp)!)_{p'} (r!)_{p'}} \langle \mu \rangle(1)_{p'} \langle \tilde{\lambda} \rangle(1)_{p'} \pmod{p}.$$

*Proof.* Since  $\delta = 1$ , we have  $\langle \tilde{\lambda} \rangle \in B_0$  and  $\sigma(\langle \lambda \rangle) = \sigma(\lambda) = \sigma(\tilde{\lambda}) = \sigma(\langle \tilde{\lambda} \rangle)$ , so that  $\mathcal{I}: \langle \lambda \rangle \mapsto \langle \tilde{\lambda} \rangle$  is sign preserving, and therefore gives a bijection between  $B$  and  $B_0$ .

Now, for any  $\langle \lambda \rangle \in B$ , we have

$$\langle \lambda \rangle(1) = 2^{\lfloor (n-m(\lambda))/2 \rfloor} \frac{n!}{\bar{h}(\lambda)} \quad \text{and} \quad \langle \tilde{\lambda} \rangle(1) = 2^{\lfloor (wp-m(\tilde{\lambda}))/2 \rfloor} \frac{(wp)!}{\bar{h}(\tilde{\lambda})}.$$

In particular, since  $B$  and  $B_0$  have a common defect group  $X$  (which is a Sylow  $p$ -subgroup of  $S(pw)$  and  $S^+(pw)$ ), the heights of  $\langle \lambda \rangle$  and  $\langle \tilde{\lambda} \rangle$  are

$$\mathfrak{h}(\langle \lambda \rangle) = \nu(|X|) - \nu(\bar{h}(\lambda)) \quad \text{and} \quad \mathfrak{h}(\langle \tilde{\lambda} \rangle) = \nu(|X|) - \nu(\bar{h}(\tilde{\lambda})) \quad \text{respectively.}$$

Now, using the notation of section 2, we have  $\bar{h}(\lambda) = \bar{h}_{\lambda,p} \bar{h}_{\lambda,p'}$  and  $\bar{h}(\tilde{\lambda}) = \bar{h}_{\tilde{\lambda},p} \bar{h}_{\tilde{\lambda},p'}$ , so that  $\nu(\bar{h}(\lambda)) = \nu(\bar{h}_{\lambda,p})$  and  $\nu(\bar{h}(\tilde{\lambda})) = \nu(\bar{h}_{\tilde{\lambda},p})$ . However, because of the bijection between bars of length divisible by  $p$  in  $\lambda$  and bars in the  $\bar{p}$ -quotient  $\lambda^{(\bar{p})}$ , we have  $\bar{h}_{\lambda,p} = p^w \bar{h}(\lambda^{(\bar{p})}) = p^w \bar{h}(\tilde{\lambda}^{(\bar{p})}) = \bar{h}_{\tilde{\lambda},p}$ , whence  $\nu(\bar{h}(\lambda)) = \nu(\bar{h}(\tilde{\lambda}))$  and  $\mathfrak{h}(\langle \lambda \rangle) = \mathfrak{h}(\langle \tilde{\lambda} \rangle)$ . This proves that  $\mathcal{I}$  is height-preserving. We also get

$$\frac{\langle \lambda \rangle(1)_{p'}}{\langle \tilde{\lambda} \rangle(1)_{p'}} = \frac{2^{\lfloor (n-m(\lambda))/2 \rfloor}}{2^{\lfloor (wp-m(\tilde{\lambda}))/2 \rfloor}} \frac{(n!)_{p'}}{((wp)!)_{p'}} \frac{(\bar{h}_{\tilde{\lambda},p})_{p'} \bar{h}_{\tilde{\lambda},p'}}{(\bar{h}_{\lambda,p})_{p'} \bar{h}_{\lambda,p'}} = \frac{2^{\lfloor (n-m(\lambda))/2 \rfloor}}{2^{\lfloor (wp-m(\tilde{\lambda}))/2 \rfloor}} \frac{(n!)_{p'}}{((wp)!)_{p'}} \frac{\bar{h}_{\tilde{\lambda},p'}}{\bar{h}_{\lambda,p'}}.$$

If we write  $\lambda^{(\bar{p})} = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{((p-1)/2)})$ , then, by [9, Corollary (2.6)], we have  $m(\lambda) = m(\lambda^{(0)}) + m(\mu) + 2a(\lambda)$  and  $m(\tilde{\lambda}) = m(\lambda^{(0)}) + m(\emptyset) + 2a(\tilde{\lambda})$ . This implies that

$$\lfloor (n-m(\lambda))/2 \rfloor = \lfloor (n-m(\lambda^{(0)})-m(\mu))/2 \rfloor - a(\lambda)$$

and

$$\lfloor (wp-m(\tilde{\lambda}))/2 \rfloor = \lfloor (wp-m(\lambda^{(0)}))/2 \rfloor - a(\tilde{\lambda}).$$

Now Proposition 2.5 gives  $\bar{h}_{\lambda,p'} \equiv \pm 2^{-a(\lambda)} \bar{h}(\mu) \pmod{p}$  and  $\bar{h}_{\tilde{\lambda},p'} \equiv \pm 2^{-a(\tilde{\lambda})} \pmod{p}$ . This yields

$$\frac{2^{\lfloor (n-m(\lambda))/2 \rfloor}}{\bar{h}_{\lambda,p'}} \equiv \pm \frac{2^{\lfloor (n-m(\lambda^{(0)})-m(\mu))/2 \rfloor}}{\bar{h}(\mu)} \pmod{p}$$

and

$$\frac{\bar{h}_{\tilde{\lambda},p'}}{2^{\lfloor (n-m(\tilde{\lambda}))/2 \rfloor}} \equiv \pm \frac{1}{2^{\lfloor (wp-m(\lambda^{(0)}))/2 \rfloor}} \pmod{p}.$$

By hypothesis, we have  $\delta = \sigma(\mu) = (-1)^{r-m(\mu)} = 1$ , so that  $r - m(\mu)$  is even. Thus  $\lfloor (n - m(\lambda^{(0)}) - m(\mu))/2 \rfloor = \lfloor (wp - m(\lambda^{(0)}))/2 \rfloor - \lfloor (r - m(\mu))/2 \rfloor$ , which in turns implies, together with the above,

$$\frac{\langle \lambda \rangle(1)_{p'}}{\langle \tilde{\lambda} \rangle(1)_{p'}} \equiv \pm \frac{(n!)_{p'}}{((wp)!)_{p'}} \frac{2^{\lfloor (r-m(\mu))/2 \rfloor}}{\bar{h}(\mu)} = \pm \frac{(n!)_{p'}}{((wp)!)_{p'}(r!)_{p'}} \langle \mu \rangle(1)_{p'} \pmod{p}.$$

□

The corresponding result when  $\delta = -1$  is given by the following

**Lemma 3.2.** *If  $\delta = -1$ , then, with the above notation,  $\lambda \mapsto \tilde{\lambda}$  induces a sign-preserving bijection  $\mathcal{I}: \langle \lambda \rangle \mapsto \langle \tilde{\lambda} \rangle_-$  between  $B$  and  $B_0$  which is also height-preserving. Furthermore, if  $\langle \tilde{\lambda} \rangle_+ \in B_0^* \subset SI(S^+(pw))$  is the dual correspondent of  $\langle \tilde{\lambda} \rangle_-$ , then*

$$\langle \lambda \rangle(1)_{p'} \equiv \pm \frac{(n!)_{p'}}{((wp)!)_{p'}(r!)_{p'}} \langle \mu \rangle(1)_{p'} \langle \tilde{\lambda} \rangle_+(1)_{p'} 2^{s(\lambda)} \pmod{p},$$

where  $s(\lambda) = 1$  if  $\sigma(\tilde{\lambda}) = -1$  and  $s(\lambda) = 0$  if  $\sigma(\tilde{\lambda}) = 1$ .

*Proof.* Since  $\delta = -1$ , we have  $\sigma(\tilde{\lambda}) = -\sigma(\lambda)$ , and  $\sigma(\langle \tilde{\lambda} \rangle_-) = -\sigma(f^-(\langle \tilde{\lambda} \rangle_-)) = -\sigma(\tilde{\lambda}) = \sigma(\langle \lambda \rangle)$ , so that  $\mathcal{I}: \langle \lambda \rangle \mapsto \langle \tilde{\lambda} \rangle_-$  is sign preserving, and therefore gives a bijection between  $B$  and  $B_0$ .

Now, for any  $\langle \lambda \rangle \in B$ , we have

$$\langle \lambda \rangle(1) = 2^{\lfloor (n-m(\lambda))/2 \rfloor} \frac{n!}{\bar{h}(\lambda)} \text{ and } \langle \tilde{\lambda} \rangle_+(1) = 2^{\lfloor (wp-m(\tilde{\lambda}))/2 \rfloor} \frac{(wp)!}{\bar{h}(\tilde{\lambda})},$$

and, by duality,

$$\langle \tilde{\lambda} \rangle_-(1) = \begin{cases} \langle \tilde{\lambda} \rangle_+(1) & \text{if } \sigma(\langle \tilde{\lambda} \rangle_-) = 1 \\ \langle \tilde{\lambda} \rangle_+(1)/2 & \text{if } \sigma(\langle \tilde{\lambda} \rangle_-) = -1 \end{cases}.$$

As in the proof of Lemma 3.1, this implies that  $\mathcal{I}$  is height-preserving.

As above, we have

$$\lfloor (n - m(\lambda))/2 \rfloor = \lfloor (n - m(\lambda^{(0)}) - m(\mu))/2 \rfloor - a(\lambda)$$

and

$$\lfloor (wp - m(\tilde{\lambda}))/2 \rfloor = \lfloor (wp - m(\lambda^{(0)}))/2 \rfloor - a(\tilde{\lambda}),$$

so that Proposition 2.5 yields

$$\frac{2^{\lfloor (n-m(\lambda))/2 \rfloor}}{\bar{h}_{\lambda,p'}} \equiv \pm \frac{2^{\lfloor (n-m(\lambda^{(0)})-m(\mu))/2 \rfloor}}{\bar{h}(\mu)} \pmod{p}$$

and

$$\frac{\bar{h}_{\tilde{\lambda},p'}}{2^{\lfloor (n-m(\tilde{\lambda}))/2 \rfloor}} \equiv \pm \frac{1}{2^{\lfloor (wp-m(\lambda^{(0)}))/2 \rfloor}} \pmod{p}.$$

However, this time, we have  $\delta = \sigma(\mu) = (-1)^{r-m(\mu)} = -1$ , so that  $r - m(\mu)$  is odd. Thus

$$\begin{aligned} \lfloor (n - m(\lambda^{(0)}) - m(\mu))/2 \rfloor &= \lfloor (wp - m(\lambda^{(0)}) + 1)/2 \rfloor - \lfloor (r - m(\mu) - 1)/2 \rfloor \\ &= \lfloor (wp - m(\lambda^{(0)}) + 1)/2 \rfloor - \lfloor (r - m(\mu))/2 \rfloor. \end{aligned}$$

Now

$$\lfloor (wp - m(\lambda^{(0)}) + 1)/2 \rfloor - \lfloor (wp - m(\lambda^{(0)}))/2 \rfloor = \begin{cases} 1 & \text{if } wp - m(\lambda^{(0)}) \text{ is odd} \\ 0 & \text{if } wp - m(\lambda^{(0)}) \text{ is even} \end{cases}.$$

But  $(-1)^{wp-m(\lambda^{(0)})} = (-1)^{wp-|\lambda^{(0)}|}(-1)^{|\lambda^{(0)}|-m(\lambda^{(0)})} = (-1)^{wp-|\lambda^{(0)}|}\sigma(\lambda^{(0)})$ , and, since  $p$  is odd,  $(-1)^{wp-|\lambda^{(0)}|} = (-1)^{w-|\lambda^{(0)}|}$ , so that  $(-1)^{wp-m(\lambda^{(0)})} = \sigma(\lambda^{(\bar{p})}) = -\sigma(\lambda) = \sigma(\tilde{\lambda})$ . This implies the result.  $\square$

We now turn to the  $p$ -local situation. As mentionned above, the defect group  $X$  of the block  $B$  can be chosen to be a Sylow  $p$ -subgroup of  $S(pw)$  and of  $S^+(pw)$ . If we let  $N_0(X) = N_{S(pw)}(X)$ , then we have

$$N_{S(n)}(X) = N_0(X) \times S(r) \text{ and } N := N_{S^+(n)}(X) = N_0(X)^+ \hat{\times} S^+(r).$$

In particular, we have, writing  $\mathcal{N} = |N_0(X)^+| = |N_{S(pw)}(X)^+| = |N_{S^+(pw)}(X)|$ ,

$$[S^+(n) : N_{S^+(n)}(X)] = \frac{2n!}{|N_0(X)^+ \hat{\times} S^+(r)|} = \frac{2n!}{(\mathcal{N}2r!)/2} = \frac{2n!}{\mathcal{N}r!}$$

and

$$[S^+(pw) : N_{S^+(pw)}(X)] = \frac{2(pw)!}{\mathcal{N}}.$$

The following result will also be useful later; note that, if  $X$  is a Sylow  $p$ -subgroup of  $S^+(pw)$ , then it is also a Sylow  $p$ -subgroup of  $S^-(pw)$  (since  $S^-(pw)$  is of index 2 in  $S^+(pw)$ , while  $p$  is odd).

**Lemma 3.3.** *If  $X$  is a Sylow  $p$ -subgroup of  $S^+(pw)$ , then*

$$[S^-(pw) : N_{S^-(pw)}(X)] = [S^+(pw) : N_{S^+(pw)}(X)]$$

and

$$[S^-(n) : N_{S^-(n)}(X)] = [S^+(n) : N_{S^+(n)}(X)].$$

*Proof.* We start by noticing that we must have

$$\frac{[S^+(pw) : N_{S^+(pw)}(X)]}{[S^-(pw) : N_{S^-(pw)}(X)]} = \frac{2}{[N_{S^+(pw)}(X) : N_{S^-(pw)}(X)]} \in \{1, 2\}$$

(since  $[N_{S^+(pw)}(X) : N_{S^-(pw)}(X)] \in \{1, 2\}$ ).

However, by Sylow's Theorems, we have  $[S^-(pw) : N_{S^-(pw)}(X)] \equiv 1 \pmod{p}$ . Thus  $[S^+(pw) : N_{S^+(pw)}(X)] = 2[S^-(pw) : N_{S^-(pw)}(X)]$  would yield  $[S^+(pw) : N_{S^+(pw)}(X)] \equiv 2 \pmod{p}$ , a contradiction,  $X$  being a Sylow  $p$ -subgroup of  $S^+(pw)$ . Hence  $[S^-(pw) : N_{S^-(pw)}(X)] = [S^+(pw) : N_{S^+(pw)}(X)]$ .

Now  $N_{S^+(n)}(X) = N_{S^+(pw)}(X) \hat{\times} S^+(r)$ . Hence

$$\begin{aligned} [S^+(n) : N_{S^+(n)}(X)] &= [S^+(n) : S^+(pw) \hat{\times} S^+(r)] [S^+(pw) \hat{\times} S^+(r) : N_{S^+(n)}(X)] \\ &= [S^+(n) : S^+(pw) \hat{\times} S^+(r)] [S^+(pw) : N_{S^+(pw)}(X)] \\ &= \frac{2n!}{(2(pw)!2r!)/2} [S^+(pw) : N_{S^+(pw)}(X)] \\ &= \frac{n!}{(pw)!r!} [S^-(pw) : N_{S^-(pw)}(X)]. \end{aligned}$$

On the other hand, we have, as sets, a disjoint union

$$N_{S^-(n)}(X) = (N_{S^-(pw)}(X) \hat{\times} S^-(r)) \bigcup (N_{S^+(pw) \setminus S^-(pw)}(X) \hat{\times} (S^+(r) \setminus S^-(r))),$$

so that  $|N_{S^-(n)}(X)| = 2|N_{S^-(pw)}(X) \hat{\times} S^-(r)|$ . We thus get

$$\begin{aligned} [S^-(n) : N_{S^-(n)}(X)] &= \frac{|S^+(n)|}{|S^-(pw) \hat{\times} S^-(r)|} \frac{|S^-(pw) \hat{\times} S^-(r)|}{2|N_{S^-(pw)}(X) \hat{\times} S^-(r)|} \\ &= \frac{n!}{((pw)!r!)/2} \frac{[S^-(pw) : N_{S^-(pw)}(X)]}{2} \\ &= \frac{n!}{(pw)!r!} [S^-(pw) : N_{S^-(pw)}(X)], \end{aligned}$$

whence the result.  $\square$

If  $\mu \succ r$  is the  $\bar{p}$ -core of  $B$ , we choose  $\gamma \in SI(S^+(r))$  such that  $f^+(\gamma) = \mu$ . We have  $\gamma = \gamma^a$  if and only if  $\delta = 1$ .

If we then denote by  $b$  the Brauer correspondent of  $B$  in  $N$ , we have (see the proof of [6, Theorem 2.2])

$$b = \{\chi \hat{\otimes} \gamma, \chi \hat{\otimes} \gamma^a \mid \chi \in SI(N_0(X)^+)\} = \{\chi \hat{\otimes} \gamma, \chi \hat{\otimes} \gamma^a \mid \chi \in \beta_0\},$$

where  $\beta_0$  is the spin block of  $N_0(X)^+$ , and thus the Brauer correspondent of the principal spin block of  $S^+(pw)$ . In particular,  $\beta_0 = b_0$  if  $\delta = 1$  and  $\beta_0 = b_0^*$  if  $\delta = -1$ .

For any  $\chi \in \beta_0$ , we have  $(\chi \hat{\otimes} \gamma)(1) = (\chi \hat{\otimes} \gamma^a)(1) = 2^{\lfloor s/2 \rfloor} \chi(1)\gamma(1)$ , where  $s$  is the number of n.s.a. characters in  $\{\chi, \gamma\}$ . If  $\delta = 1$ , we therefore get  $s = 0$  (if  $\chi^a = \chi$ ) or  $s = 1$  (if  $\chi^a \neq \chi$ ), so that  $\lfloor s/2 \rfloor = 0$  and  $(\chi \hat{\otimes} \gamma)(1) = \chi(1)\gamma(1)$ . If  $\delta = -1$ , we have  $s = 1$  and  $\lfloor s/2 \rfloor = 0$  if  $\chi^a = \chi$ , and  $s = 2$  and  $\lfloor s/2 \rfloor = 1$  if  $\chi^a \neq \chi$ , so that

$$(\chi \hat{\otimes} \gamma)(1) = (\chi \hat{\otimes} \gamma^a)(1) = \begin{cases} \chi(1)\gamma(1) & \text{if } \chi^a = \chi \\ 2\chi(1)\gamma(1) & \text{if } \chi^a \neq \chi \end{cases}.$$

**3.2. Reduction Theorem.** We can now prove the main result of this section:

**Theorem 3.4.** *Let  $B$  be a spin block of  $S^\varepsilon(n)$  of weight  $w = w(B) > 0$  and sign  $\delta = \delta(B)$  and let  $b$  be its Brauer correspondent in  $N_{S^\varepsilon(n)}(X)$ , where  $X$  is a defect group of  $B$ . Suppose the Isaacs-Navarro Conjecture holds for the principal spin block of  $S^+(pw)$  via a sign-preserving bijection. Then it also holds for  $B$ .*

*Proof.* We first suppose  $\varepsilon = 1$ .

We use the same notation as in Section 3.1. Let  $B_0$  be the principal spin block of  $S^\delta(pw)$  and  $b_0$  be its Brauer correspondent. Let  $\mu \succ r = n - wp$  be the  $\bar{p}$ -core of  $B$ , and  $\gamma \in SI(S^+(r))$  such that  $f^+(\gamma) = \mu$ . If  $\lambda$  is a bar-partition of  $n$  with  $\bar{p}$ -core  $\mu$  and  $\bar{p}$ -quotient  $\lambda^{(\bar{p})}$ , let  $\tilde{\lambda}$  be the bar-partition of  $wp$  with empty  $\bar{p}$ -core and  $\bar{p}$ -quotient  $\lambda^{(\bar{p})}$ .

Suppose furthermore that  $\delta = 1$ . Then, by Lemma 3.1,  $\lambda \mapsto \tilde{\lambda}$  induces a sign-preserving bijection  $\mathcal{I}$  between  $B$  and  $B_0$  which is also height-preserving, and

$$\langle \lambda \rangle(1)_{p'} \equiv \pm \frac{(n!)_{p'}}{((wp!)_{p'}(r!)_{p'})} \gamma(1)_{p'} \langle \tilde{\lambda} \rangle(1)_{p'} \pmod{p}.$$

Now let  $c = [S^+(pw) : N_{S^+(pw)}(X)]_{p'}$ , and let  $\varphi: M(B_0) \rightarrow M(b_0)$  be a bijection such that, for each  $k$  such that  $(p, k) = 1$ , we have  $M_k(b_0) = \varphi(M_{ck}(B_0))$  (such a  $\varphi$  exists by hypothesis). We have  $b = \{\chi \hat{\otimes} \gamma \mid \chi \in b_0\}$ , and, by [6, Proposition 1.2],  $\chi \hat{\otimes} \gamma = \psi \hat{\otimes} \gamma$  if and only if  $\psi \in \{\chi, \chi^a\}$  and  $(\sigma(\chi)\sigma(\gamma) = \sigma(\chi) = 1)$  or  $(\sigma(\chi) = -1$

and  $\chi = \psi$ ), i.e.  $\chi \hat{\otimes} \gamma = \psi \hat{\otimes} \gamma$  if and only if  $\psi = \chi$ . Thus, by the results of Section 3.1,  $\chi \mapsto \chi \hat{\otimes} \gamma$  is a height-preserving bijection between  $b_0$  and  $b$ . Hence

$$\Phi: \begin{cases} M(B) & \longrightarrow M(b) \\ \langle \lambda \rangle & \mapsto \varphi(\langle \tilde{\lambda} \rangle) \hat{\otimes} \gamma \end{cases}$$

is a (height-preserving) sign-preserving bijection.

Now, if  $\langle \lambda \rangle \in M(B)$ , then  $\Phi(\langle \lambda \rangle)(1) = (\varphi(\langle \tilde{\lambda} \rangle))(1)\gamma(1)$ , so that

$$\begin{aligned} \Phi(\langle \lambda \rangle)(1)_{p'} &= (\varphi(\langle \tilde{\lambda} \rangle))(1)_{p'} \gamma(1)_{p'} \\ &\equiv \pm \frac{\langle \tilde{\lambda} \rangle(1)_{p'}}{c} \gamma(1)_{p'} \pmod{p} \quad (\text{by definition of } \varphi) \\ &\equiv \pm \frac{1}{c} \left( \frac{r!(pw)!}{n!} \right)_{p'} \langle \lambda \rangle(1)_{p'} \pmod{p}, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{c} \left( \frac{r!(pw)!}{n!} \right)_{p'} &= \frac{1}{[S^+(pw): N_{S^+(pw)}(X)]_{p'}} \left( \frac{r!(pw)!}{n!} \right)_{p'} \\ &= \left( \frac{\mathcal{N}}{2(pw)!} \right)_{p'} \left( \frac{r!(pw)!}{n!} \right)_{p'} = \left( \frac{\mathcal{N}r!}{2n!} \right)_{p'} \\ &= \frac{1}{[S^+(n): N_{S^+(n)}(X)]_{p'}}, \end{aligned}$$

whence we finally get

$$\Phi(\langle \lambda \rangle)(1)_{p'} \equiv \pm \frac{\langle \lambda \rangle(1)_{p'}}{[S^+(n): N_{S^+(n)}(X)]_{p'}} \pmod{p},$$

i.e.  $\Phi$  is an Isaacs-Navarro bijection between  $B$  and  $b$ .

Suppose now that  $\delta = -1$ . Then  $B_0$  is the principal spin block of  $S^-(pw)$ , its dual  $B_0^*$  is the principal spin block of  $S^+(pw)$ , and  $b_0^*$  is the Brauer correspondent of  $B_0^*$ . Writing  $D_+$  for the set of s.a. characters in  $D$  and  $D_-$  for the set of pairs of n.s.a. characters in  $D$  (so that  $|M_k(B_0)| = |M_k(B_0)_+| + 2|M_k(B_0)_-|$ ), we thus have, for each  $k$  such that  $(p, k) = 1$ , the following equalities:

$$|M_k(B_0)_+| = |M_k(B_0)_-| = |M_{k/c}(b_0^*)_+|$$

and

$$|M_k(B_0)_-| = |M_{2k}(B_0^*)_+| = |M_{2k/c}(b_0^*)_+|,$$

where  $c = [S^+(pw): N_{S^+(pw)}(X)]_{p'} = [S^-(pw): N_{S^-(pw)}(X)]_{p'}$  (by Lemma 3.3).

On the other hand, we have  $b = \{\chi \hat{\otimes} \gamma, \chi \hat{\otimes} \gamma^a \mid \chi \in \beta_0 = b_0^*\}$ . For any  $\chi, \psi \in \beta_0$  and  $\gamma_1, \gamma_2 \in \{\gamma, \gamma^a\}$ , we have  $\chi \hat{\otimes} \gamma_1 = \psi \hat{\otimes} \gamma_2$  if and only if  $\chi$  and  $\psi$  are associate and

$$\begin{cases} \sigma(\chi)\sigma(\gamma_1) = -\sigma(\chi) = 1 \\ \text{or} \\ \sigma(\chi)\sigma(\gamma_1) = -\sigma(\chi) = -1 \text{ and } [(\chi = \psi, \gamma_1 = \gamma_2) \text{ or } (\chi \neq \psi, \gamma_1 \neq \gamma_2)] \end{cases}$$

Hence, if  $\chi \in \beta_{0+}$ , then we get two irreducible characters,  $\chi \hat{\otimes} \gamma$  and  $\chi \hat{\otimes} \gamma^a$ , while, if  $\chi \in \beta_{0-}$ , then we get one irreducible character,  $\chi \hat{\otimes} \gamma = \chi^a \hat{\otimes} \gamma = \chi \hat{\otimes} \gamma^a = \chi \hat{\otimes} \gamma^a$ . Note that  $\chi \mapsto \chi \hat{\otimes} \gamma$  and  $\chi \mapsto \chi \hat{\otimes} \gamma^a$  are height preserving. Using the equalities

above, as well as Lemma 3.2, we obtain the following height-preserving and sign-preserving bijection:

$$\Phi: \begin{cases} M(B) \xrightarrow{\dot{\rightarrow}} M(B_0) \xrightarrow{\dot{\rightarrow}} M(B_0^*) \xrightarrow{\dot{\rightarrow}} M(b_0^*) \xrightarrow{\dot{\rightarrow}} M(b) \\ \langle \lambda \rangle \mapsto \langle \tilde{\lambda} \rangle_- \mapsto \langle \tilde{\lambda} \rangle_+ \mapsto \varphi(\langle \tilde{\lambda} \rangle_+) \mapsto \varphi(\langle \tilde{\lambda} \rangle_+) \hat{\otimes} \langle \mu \rangle \end{cases},$$

where, as before,  $\varphi$  is the (sign-preserving) Isaacs-Navarro bijection we supposed exists between  $M(B_0^*)$  and  $M(b_0^*)$ , and  $\dot{\rightarrow}$  (respectively  $\dot{\leftarrow}$ ) denotes a sign-preserving (respectively sign-inversing) bijection.

Now, by hypothesis,  $\langle \tilde{\lambda} \rangle_+(1)_{p'} \equiv c\varphi(\langle \tilde{\lambda} \rangle_+)(1)_{p'} \pmod{p}$ , so that, by Lemma 3.2, we obtain

$$\langle \lambda \rangle(1)_{p'} \equiv \left( \frac{n!}{(wp)!r!} \right)_{p'} c\varphi(\langle \tilde{\lambda} \rangle_+)(1)_{p'} \langle \mu \rangle(1)_{p'} 2^{s(\lambda)} \pmod{p},$$

and, as in the case  $\delta = 1$ , we have  $\left( \frac{n!}{(wp)!r!} \right)_{p'} c = [S^+(n): N_{S^+(n)}(X)]_{p'}$ . Finally, since  $\sigma(\tilde{\lambda}) = 1 \iff \sigma(\langle \tilde{\lambda} \rangle_+) = 1 \iff \langle \tilde{\lambda} \rangle_+$  is s.a.  $\iff \varphi(\langle \tilde{\lambda} \rangle_+)$  is s.a., we get

$$\begin{aligned} \varphi(\langle \tilde{\lambda} \rangle_+)(1)_{p'} \langle \mu \rangle(1)_{p'} 2^{s(\lambda)} &= \begin{cases} 2\varphi(\langle \tilde{\lambda} \rangle_+)(1)_{p'} \langle \mu \rangle(1)_{p'} & \text{if } \sigma(\tilde{\lambda}) = -1 \\ \varphi(\langle \tilde{\lambda} \rangle_+)(1)_{p'} \langle \mu \rangle(1)_{p'} & \text{if } \sigma(\tilde{\lambda}) = 1 \end{cases} \\ &= (\varphi(\langle \tilde{\lambda} \rangle_+) \hat{\otimes} \langle \mu \rangle)(1)_{p'} \\ &= \Phi(\langle \lambda \rangle)(1)_{p'}, \end{aligned}$$

whence  $\langle \lambda \rangle(1)_{p'} \equiv [S^+(n): N_{S^+(n)}(X)]_{p'} \Phi(\langle \lambda \rangle)(1)_{p'} \pmod{p}$ , i.e.  $\Phi$  is a (sign-preserving) Isaacs-Navarro bijection between  $M(B)$  and  $M(b)$ .

We now suppose  $\varepsilon = -1$ .

In this case,  $B$  is a spin block of  $S^-(n)$  and  $b$  is its Brauer correspondent in  $N_{S^-(n)}(X)$ . Thus  $B^*$  is a spin block of  $S^+(n)$ , and, by [6, Lemma 2.3] (which is due to H. Blau), the dual  $b^*$  of  $b$  is the Brauer correspondent of  $B^*$ . By the case  $\varepsilon = 1$ , there exists a sign-preserving Isaacs-Navarro bijection  $\varphi: M(B^*) \rightarrow M(b^*)$ . We define the sign-preserving bijection

$$\Phi: \begin{cases} M(B) \longrightarrow M(b) \\ \langle \lambda \rangle \mapsto (\varphi(\langle \lambda \rangle^*))^* \end{cases}$$

By Lemma 3.3, we have  $c = [S^-(n): N_{S^-(n)}(X)]_{p'} = [S^+(n): N_{S^+(n)}(X)]_{p'}$ , and, for each  $k$  such that  $(p, k) = 1$ , we have

$$|M_{ck}(B)_+| = |M_{ck}(B^*)_+| = |M_k(b^*)_+| = |M_k(b)_+|$$

and

$$|M_{ck}(B)_-| = |M_{2ck}(B^*)_+| = |M_{2k}(b^*)_+| = |M_k(b)_-|,$$

whence  $|M_{ck}(B)| = |M_{ck}(B)_+| + 2|M_{ck}(B)_-| = |M_k(b)|$ . □

#### 4. PRINCIPAL BLOCK

By Theorem 3.4, it is now sufficient to prove that the Isaacs-Navarro Conjecture holds for the principal spin block of  $S^+(pw)$  via a sign-preserving bijection. Throughout this section, we therefore consider the following situation. We take  $G = S^+(pw)$  (where  $w \geq 1$  is an integer),  $B$  the principal spin block of  $G$ , and  $b$  the Brauer correspondent of  $B$ . Hence  $b$  is the principal spin block of  $N_G(X)$  for some  $X \in Syl_p(G)$ .

**4.1. Spin characters of height 0 of the normalizer.** The normalizer  $N^+ = N_G(X)$  and its irreducible spin characters are described in Sections 3 and 4 of [6]. Let  $pw = \sum_{i=1}^k t_i p^i$  be the  $p$ -adic decomposition of  $pw$ . We then have  $N^+ = [N_1 \wr S(t_1)]^+ \hat{\times} \cdots \hat{\times} [N_k \wr S(t_k)]^+$ , where, for each  $1 \leq i \leq k$ ,  $N_i = N_{S(p^i)}(X_i)$  for some  $X_i \in Syl_p(S(p^i))$ .

Now fix  $1 \leq i \leq k$ , and let  $e_i = (p^i - 1)/2$ . Then  $H_i^+ = (N_i \wr S(t_i))^+ = M_i^+ S_{t_i}^+$ , where  $M_i^+ = N_i^{(1)+} \hat{\times} \cdots \hat{\times} N_i^{(e_i)+} \triangleleft H_i^+$  and  $S_{t_i} \cong \Delta_{p^i} S(t_i) \subset S(p^i t_i)$ , and where

$$S_{t_i}^+ \cong \begin{cases} \hat{S}(t_i) & \text{if } p^i \equiv 1 \pmod{4} \\ \tilde{S}(t_i) & \text{if } p^i \equiv -1 \pmod{4} \end{cases}.$$

By [6, Proposition 3.9],  $N_i^+$  has one s.a. spin character  $\zeta_0$  of degree  $(p-1)^i$ , and  $e_i = (p^i - 1)/2$  pairs of n.s.a. spin characters  $\{\zeta_1, \zeta_1^a, \dots, \zeta_{e_i}, \zeta_{e_i}^a\}$  of degree 1.

Let  $\mathcal{A}_i = \{(t_i^{(0)}, t_i^{(1)}, \dots, t_i^{(e_i)}) \mid t_i^{(j)} \in \mathbb{N} \cup \{0\}, \sum_{j=0}^{e_i} t_i^{(j)} = t_i\}$ . Then, by [6, Proposition 3.12], a complete set of representatives for the  $S_{t_i}^+$ -conjugacy classes in  $SI_0(M_i^+)$  is given by

$$\mathcal{R} = \{\theta_s \mid s \in \mathcal{A}_i\} \cup \{\theta_s^a \mid s = (t_i^{(0)}, t_i^{(1)}, \dots, t_i^{(e_i)}) \in \mathcal{A}_i, t_i - t_i^{(0)} \text{ odd}, t_i^{(0)} \leq 1\},$$

where  $\theta_s = \theta_0 \hat{\otimes} \theta_1 \hat{\otimes} \cdots \hat{\otimes} \theta_{e_i}$ , with  $\theta_j = \zeta_j \hat{\otimes} \cdots \hat{\otimes} \zeta_j$  ( $t_i^{(j)}$  factors). Also, the inertial subgroup  $T_i^+ = I_{H_i^+}(\theta_s)$  of  $\theta_s$  in  $H_i^+$  satisfies

$$T_i^+/M_i^+ \cong \begin{cases} A(t_i^{(0)}) \times S(t_i^{(1)}) \times \cdots \times S(t_i^{(e_i)}) & \text{if } t_i - t_i^{(0)} \text{ is odd} \\ S(t_i^{(0)}) \times S(t_i^{(1)}) \times \cdots \times S(t_i^{(e_i)}) & \text{if } t_i - t_i^{(0)} \text{ is even} \end{cases}.$$

We can now describe how to induce each  $\theta_j$  from  $M_i^{(j)+} = (N_i^+)^{\hat{\times} t_i^{(j)}}$  to the corresponding factor  $T_i^{(j)+}$  of its inertial subgroup.

**Proposition 4.1.** [6, Proposition 4.4] *If  $\zeta$  is a n.s.a. linear representation of  $N_i^+$ , then  $\theta_j = \zeta^{t_i^{(j)}} = \zeta \hat{\otimes} \cdots \hat{\otimes} \zeta \in \text{Irr}(M_i^{(j)+})$  can be extended to a negative representation  $D_\zeta \in \text{Irr}(T_i^{(j)+})$ , and every irreducible constituent  $V$  of  $\theta \uparrow_{T_i^{(j)+}}$  is of the form  $V = D_\zeta \otimes R$ , where  $R$  is an irreducible representation of  $T_i^{(j)+}/M_i^{(j)+} \cong S(t_i^{(j)})$ . If  $t_i^{(j)}$  is odd, then every irreducible constituent  $V$  of  $\theta \uparrow_{T_i^{(j)+}}$  is n.s.a., and, if  $t_i^{(j)}$  is even, then every irreducible constituent  $V$  of  $\theta \uparrow_{T_i^{(j)+}}$  is s.a..*

In the above notation, if  $\psi$  is the character of  $V = D_\zeta \otimes R$ , and if  $R$  has character  $\chi_\lambda \in \text{Irr}(S(t_i^{(j)}))$ , then  $\psi(1) = \zeta^{t_i^{(j)}}(1)\chi_\lambda(1)$ . Also, since  $\zeta$  is n.s.a., we have  $\zeta^{t_i^{(j)}}(1) = 2^{\lfloor t_i^{(j)}/2 \rfloor} \zeta(1)^{t_i^{(j)}} = 2^{\lfloor t_i^{(j)}/2 \rfloor} \chi_\lambda(1)$ , and  $\psi(1) = 2^{\lfloor t_i^{(j)}/2 \rfloor} \chi_\lambda(1)$ . Finally,  $\psi$  is s.a. if and only if  $t_i^{(j)}$  is even.

**Proposition 4.2.** [6, Proposition 4.8] *Let  $t_i^{(0)} \geq 4$ , and let  $D$  be the s.a. spin representation of  $N_i^+$  with degree  $(p-1)^i$ . Then  $D^{t_i^{(0)}} = D \hat{\otimes} \cdots \hat{\otimes} D \in \text{Irr}(M_i^{(0)+})$  can neither be extended to an irreducible representation of  $T_i^{(0)-} = M_i^{(0)+} A_{t_i^{(0)}}^+$  nor to one of  $T_i^{(0)+} = M_i^{(0)+} S_{t_i^{(0)}}^+$ . Furthermore, every irreducible constituent  $V$  of  $D^{t_i^{(0)}} \uparrow_{T_i^{(0)+}}$  is of the form  $V = D^{t_i^{(0)}} \otimes S$ , where  $S$  is an irreducible spin representation of  $S_{t_i^{(0)}}^+$ , and every irreducible constituent  $V$  of  $D^{t_i^{(0)}} \uparrow_{T_i^{(0)-}}$  is of the form*

$V = D^{t_i^{(0)}} \otimes S$ , where  $S$  is an irreducible spin representation of  $A_{t_i^{(0)}}^+$ . In each case,  $V$  is s.a. if and only if  $S$  is s.a..

In this notation, if  $\psi$  is the character of  $V$ , and if  $S$  has character  $\chi_S$ , then  $\psi(1) = \zeta_0^{t_i^{(0)}}(1)\chi_S(1)$ . And, since  $\zeta_0$  is s.a., we have  $\psi(1) = (p-1)^{it_i^{(0)}}\chi_S(1)$ .

We can now describe all the characters of height 0 in  $b$ . Recall that these are exactly the spin characters with  $p$ -degree in  $N^+$ . Still writing  $pw = \sum_{i=1}^k t_i p^i$  the  $p$ -adic decomposition of  $pw$ , Olsson proved in [10] that, for any sign  $\sigma$ ,

$$|M(b)_\sigma| = \sum_{\{(\sigma_1, \dots, \sigma_k)\}} \prod_{i=1}^k q^{\sigma_i}(\bar{p}^i, t_i),$$

where  $(\sigma_1, \dots, \sigma_k)$  runs through all  $k$ -tuples of signs satisfying  $\sigma_1 \dots \sigma_k = \sigma$ , and where  $q^{\sigma_i}(\bar{p}^i, t_i)$  denotes the number of all  $\bar{p}^i$ -quotients with sign  $\sigma_i$  and weight  $t_i$ .

The correspondence goes as follows. For each  $1 \leq i \leq k$ , pick  $\mathbf{s}_i \in \mathcal{A}_i$  and the corresponding  $\theta_{\mathbf{s}_i} \in SI_0(M_{\mathbf{s}_i}^+)$  (where, if  $\mathbf{s}_i = (t_i^{(0)}, t_i^{(1)}, \dots, t_i^{(e_i)})$ , then  $M_{\mathbf{s}_i}^+ = (N_i^{\hat{\times} t_i^{(0)}})^+ \hat{\times} \dots \hat{\times} (N_i^{\hat{\times} t_i^{(e_i)}})^+$ ). Inducing  $\theta_{\mathbf{s}_i}$  (or  $\theta_{\mathbf{s}_i} + \theta_{\mathbf{s}_i}^a$  if  $t_i - t_i^{(0)}$  is odd and  $t_i^{(0)} \leq 1$ ) to its inertial subgroup  $T_i^+$ , we obtain s.a. irreducible constituents and pairs of n.s.a. irreducible constituents described by Propositions 4.1 and 4.2 and labeled by the  $\bar{p}^i$ -quotients of weight  $t_i$ : if  $Q_i = (\lambda_i^{(0)}, \lambda_i^{(1)}, \dots, \lambda_i^{(e_i)})$  is a  $\bar{p}^i$ -quotient of weight  $t_i$ , then  $\langle \Psi_{Q_i} \rangle = \langle \psi_i^{(0)} \rangle \hat{\times} \langle \psi_i^{(1)} \rangle \hat{\times} \dots \hat{\times} \langle \psi_i^{(e_i)} \rangle \in SI_0(T_i^+)$ . Also, by Propositions 4.1 and 4.2,

- For  $1 \leq j \leq e_i$ ,  $\psi_i^{(j)}(1) = 2^{\lfloor t_i^{(j)}/2 \rfloor} \chi_{\lambda_i^{(j)}}(1)$  (with  $\chi_{\lambda_i^{(j)}} \in \text{Irr}(S(t_i^{(j)}))$ ) and  $\psi_i^{(j)}$  is s.a. if and only if  $t_i^{(j)}$  is even.
- $\psi_i^{(0)}(1) = (p-1)^{it_i^{(0)}} \chi_{\lambda_i^{(0)}}(1)$  (with  $\chi_{\lambda_i^{(0)}} \in SI(S_{t_i^{(0)}}^+)$  if  $t_i - t_i^{(0)}$  is even and  $\chi_{\lambda_i^{(0)}} \in SI(A_{t_i^{(0)}}^+)$  if  $t_i - t_i^{(0)}$  is odd) and  $\psi_i^{(0)}$  is s.a. if and only if  $\chi_{\lambda_i^{(0)}}$  is s.a..

Finally,  $\langle \Psi_{Q_i} \rangle(1) = 2^{\lfloor S_i/2 \rfloor} \langle \psi_i^{(0)} \rangle(1) \langle \psi_i^{(1)} \rangle(1) \dots \langle \psi_i^{(e_i)} \rangle(1)$ , where  $S_i$  is the number of (pairs of) n.s.a. characters in  $\{\langle \psi_i^{(0)} \rangle, \langle \psi_i^{(1)} \rangle, \dots, \langle \psi_i^{(e_i)} \rangle\}$ .

Inducing to  $H_i^+ = [N_i \wr S(t_i)]^+$ , we obtain a s.a. irreducible spin character, or a pair of associate (n.s.a.) spin characters,  $\langle Q_i \rangle$ , labeled by  $Q_i$ .

Given the structure of  $T_i^+$ , we see that  $\sigma(\langle Q_i \rangle) = (-1)^{t_i - t_i^{(0)}} \sigma(\Psi_{Q_i})$ . However, we have  $\sigma(\Psi_{Q_i}) = \sigma(\psi_i^{(0)}) \sigma(\psi_i^{(1)}) \dots \sigma(\psi_i^{(e_i)})$ . Also, for  $1 \leq j \leq e_i$ , we have  $\sigma(\psi_i^{(j)}) = (-1)^{t_i^{(j)}}$ , and  $\sigma(\psi_i^{(0)}) = \sigma(\chi_{\lambda_i^{(0)}}) = \sigma(\lambda_i^{(0)}) (-1)^{t_i - t_i^{(0)}}$ , so that  $\sigma(\Psi_{Q_i}) = \sigma(\lambda_i^{(0)})$  (since  $\sum_{j=1}^{e_i} t_i^{(j)} = t_i - t_i^{(0)}$ ) and  $\sigma(\langle Q_i \rangle) = (-1)^{t_i - t_i^{(0)}} \sigma(\lambda_i^{(0)}) = \sigma(Q_i)$ . Note that, writing  $m_i^{(0)}$  for the number of (non-zero) parts in  $\lambda_i^{(0)}$ , we have  $\sigma(\lambda_i^{(0)}) = (-1)^{t_i^{(0)} - m_i^{(0)}}$ , so that  $\sigma(\langle Q_i \rangle) = (-1)^{t_i - m_i^{(0)}}$ , and  $\langle Q_i \rangle$  is s.a. if and only if  $t_i - m_i^{(0)}$  is even.

Also, we have  $\langle Q_i \rangle(1) = (|H_i^+|/|T_i^+|) \langle \Psi_{Q_i} \rangle(1)$ , unless  $\chi_{\lambda_i^{(0)}}$  is a s.a. irreducible spin character of  $A_{t_i^{(0)}}^+$  (i.e.  $t_i - t_i^{(0)}$  is odd and  $\chi_{\lambda_i^{(0)}}$  is s.a.), in which case  $\langle Q_i \rangle(1) = (|H_i^+|/|T_i^+|) \langle \Psi_{Q_i} \rangle(1)/2$ .

Finally, the irreducible characters of height 0 in  $b$  are parametrized by the sequences  $(Q_1, \dots, Q_k)$ , where  $Q_i$  is a  $\bar{p}^i$ -quotient of weight  $t_i$ . We have

$\langle (Q_1, \dots, Q_k) \rangle = \langle Q_1 \rangle \hat{\otimes} \dots \hat{\otimes} \langle Q_k \rangle$ , and  $\langle (Q_1, \dots, Q_k) \rangle(1) = 2^{\lfloor s/2 \rfloor} \prod_{i=1}^k \langle Q_i \rangle(1)$ , where  $s$  is the number of (pairs of) n.s.a. characters in  $\{\langle Q_1 \rangle, \dots, \langle Q_k \rangle\}$ . By the above remark on the sign of  $\langle Q_i \rangle$ , we see that  $s = |\{1 \leq i \leq k; t_i - m_i^{(0)} \text{ odd}\}|$ .

**Proposition 4.3.** *With the above notation, we have*

$$\frac{|N_G(X)|_{p'}}{\langle (Q_1, \dots, Q_k) \rangle(1)_{p'}} \equiv \pm \frac{2}{2^{\lfloor s/2 \rfloor}} \prod_{i=1}^k \frac{1}{2^{\lfloor (t_i - m_i^{(0)})/2 \rfloor}} \bar{h}(Q_i) \pmod{p},$$

where  $s = |\{1 \leq i \leq k; t_i - m_i^{(0)} \text{ odd}\}|$ , and, for each  $1 \leq i \leq k$ ,  $\bar{h}(Q_i)$  is the product of all bar-lengths in  $Q_i$ .

*Proof.* We have

$$\frac{|N_G(X)|_{p'}}{\langle (Q_1, \dots, Q_k) \rangle(1)_{p'}} = \frac{\prod_{i=1}^k |H_i^+|}{2^{k-1} \langle (Q_1, \dots, Q_k) \rangle(1)_{p'}} = \frac{\prod_{i=1}^k |H_i^+|}{2^{k-1} 2^{\lfloor s/2 \rfloor} \prod_{i=1}^k \langle Q_i \rangle(1)_{p'}}.$$

This gives

$$\frac{|N_G(X)|_{p'}}{\langle (Q_1, \dots, Q_k) \rangle(1)_{p'}} = \frac{1}{2^{k-1}} \frac{1}{2^{\lfloor s/2 \rfloor}} D_1^{(+1)} D_0^{(+1)} D_1^{(-1)} D_0^{(-1)},$$

where, for  $\varepsilon \in \{+1, -1\}$  and  $a \in \{0, 1\}$ ,

$$D_a^{(\varepsilon)} = \prod_{\substack{1 \leq i \leq k \\ t_i - t_i^{(0)} \equiv a \pmod{2} \\ \sigma(\chi_{\lambda_i^{(0)}}) = \varepsilon}} \frac{|H_i^+|}{\langle Q_i \rangle(1)_{p'}}.$$

Now we have, whenever  $(\varepsilon, a) \in \{(+1, 0), (-1, 1), (-1, 0)\}$ ,

$$D_a^{(\varepsilon)} = \prod_{\substack{1 \leq i \leq k \\ t_i - t_i^{(0)} \equiv a \pmod{2} \\ \sigma(\chi_{\lambda_i^{(0)}}) = \varepsilon}} \frac{|T_i^+|}{\langle \Psi_{Q_i} \rangle(1)_{p'}} = \prod_{\substack{1 \leq i \leq k \\ t_i - t_i^{(0)} \equiv a \pmod{2} \\ \sigma(\chi_{\lambda_i^{(0)}}) = \varepsilon}} \frac{|T_i^+|}{2^{\lfloor S_i/2 \rfloor} \prod_{j=0}^{e_i} \psi_i^{(j)}(1)},$$

while

$$D_1^{(+1)} = \prod_{\substack{1 \leq i \leq k \\ t_i - t_i^{(0)} \text{ odd} \\ \sigma(\chi_{\lambda_i^{(0)}}) = 1}} \frac{|T_i^+|}{\langle \Psi_{Q_i} \rangle(1)_{p'}/2} = \prod_{\substack{1 \leq i \leq k \\ t_i - t_i^{(0)} \text{ odd} \\ \sigma(\chi_{\lambda_i^{(0)}}) = 1}} \frac{|T_i^+|}{2^{\lfloor S_i/2 \rfloor - 1} \prod_{j=0}^{e_i} \psi_i^{(j)}(1)},$$

where  $S_i$  is the number of (pairs of) n.s.a. characters in  $\{\langle \psi_i^{(0)} \rangle, \dots, \langle \psi_i^{(e_i)} \rangle\}$ .

For each  $1 \leq i \leq k$ , we have  $|T_i^+|_{p'} = |T_i^+|/M_i^+|_{p'}|M_i^+|_{p'}$ . Also,  $M_i^+ \cong (N_i^+)^{\hat{\times} t_i}$ , so that  $|M_i^+| = \frac{|N_i^+|^{t_i}}{2^{t_i-1}} = 2|N_i|^{t_i}$ . But  $N_i = N_{S(p^i)}(X_i)$  for some  $X_i \in \text{Syl}_p(S(p^i))$ ; thus  $|N_i| = |X_i| \cdot |N_i/X_i|$ , and we have  $N_i/X_i = K_i \cong (\mathbb{Z}/(p-1)\mathbb{Z})^i$  (see [6, page 89]). Hence  $|N_i|_{p'} = (p-1)^i$ ,  $|M_i^+|_{p'} = 2(p-1)^{it_i}$ , and

$$|T_i^+|_{p'} \equiv \begin{cases} (-1)^{it_i} t_i^{(0)}! t_i^{(1)}! \dots t_i^{(e_i)}! \pmod{p} & \text{if } t_i - t_i^{(0)} \text{ is odd} \\ 2(-1)^{it_i} t_i^{(0)}! t_i^{(1)}! \dots t_i^{(e_i)}! \pmod{p} & \text{if } t_i - t_i^{(0)} \text{ is even} \end{cases}.$$

Now fix  $1 \leq i \leq k$ , and let  $\{1, \dots, e_i\} = I_1^{(i)} \cup I_2^{(i)}$ , where

$$I_1^{(i)} = \{j \in \{1, \dots, e_i\} \mid t_i^{(j)} = 2k_i^{(j)} + 1 (k_i^{(j)} \in \mathbb{N} \cup \{0\})\}$$

and

$$I_2^{(i)} = \{j \in \{1, \dots, e_i\} \mid t_i^{(j)} = 2k_i^{(j)} (k_i^{(j)} \in \mathbb{N})\}.$$

We obtain

$$\prod_{j=1}^{e_i} \psi_i^{(j)}(1) = \prod_{j \in I_1^{(i)}} 2^{k_i^{(j)}} \chi_{\lambda_i^{(j)}}(1) \prod_{j \in I_2^{(i)}} 2^{k_i^{(j)}} \chi_{\lambda_i^{(j)}}(1) = 2^{\sum_{j=1}^{e_i} k_i^{(j)}} \prod_{j=1}^{e_i} \chi_{\lambda_i^{(j)}}(1).$$

Note that  $S_i = |I_1^{(i)}|$  if  $\psi_i^{(0)}$  is s.a., while  $S_i = |I_1^{(i)}| + 1$  if  $\psi_i^{(0)}$  is n.s.a.. We thus have

$$\prod_{j=1}^{\lfloor S_i/2 \rfloor} \psi_i^{(j)}(1) = 2^{\sum_{j=1}^{e_i} k_i^{(j)} + \lfloor S_i/2 \rfloor} \prod_{j=1}^{e_i} \chi_{\lambda_i^{(j)}}(1) = 2^{\lfloor \sum_{j=1}^{e_i} k_i^{(j)} + S_i/2 \rfloor} \prod_{j=1}^{e_i} \chi_{\lambda_i^{(j)}}(1),$$

and

$$\begin{aligned} \lfloor \sum_{j=1}^{e_i} k_i^{(j)} + \frac{S_i}{2} \rfloor &= \lfloor \sum_{j \in I_1^{(i)}} \frac{t_i^{(j)} - 1}{2} + \sum_{j \in I_2^{(i)}} \frac{t_i^{(j)}}{2} + \frac{S_i}{2} \rfloor \\ &= \begin{cases} \lfloor \sum_{j \in I_1^{(i)}} \frac{t_i^{(j)}}{2} + \sum_{j \in I_2^{(i)}} \frac{t_i^{(j)}}{2} \rfloor = \lfloor \frac{t_i - t_i^{(0)}}{2} \rfloor & \text{if } \sigma(\psi_i^{(0)}) = 1 \\ \lfloor \sum_{j \in I_1^{(i)}} \frac{t_i^{(j)}}{2} + \sum_{j \in I_2^{(i)}} \frac{t_i^{(j)}}{2} + \frac{1}{2} \rfloor = \lfloor \frac{t_i - t_i^{(0)} + 1}{2} \rfloor & \text{if } \sigma(\psi_i^{(0)}) = -1 \end{cases} \end{aligned}$$

We can now compute  $D_1^{(+1)}$ ,  $D_0^{(+1)}$ ,  $D_1^{(-1)}$  and  $D_0^{(-1)}$ . Take any  $1 \leq i \leq k$ , and first suppose that  $t_i - t_i^{(0)}$  is odd and  $\sigma(\chi_{\lambda_i^{(0)}}) = 1$ . Then  $\psi_i^{(0)}$  is n.s.a.,  $S_i = |I_1^{(i)}| + 1$ ,  $t_i - m_i^{(0)}$  is even and

$$\chi_{\lambda_i^{(0)}}(1) = 2^{\lfloor \frac{t_i^{(0)} - m_i^{(0)}}{2} \rfloor} \frac{t_i^{(0)}!}{\bar{h}(\lambda_i^{(0)})}.$$

Finally,  $\lfloor \frac{t_i - t_i^{(0)} + 1}{2} \rfloor = \lfloor \frac{t_i - t_i^{(0)}}{2} \rfloor + 1$  (since  $t_i - t_i^{(0)}$  is odd). We therefore get

$$\begin{aligned} \frac{|T_i^+|}{2^{\lfloor S_i/2 \rfloor - 1} \prod_{j=0}^{e_i} \psi_i^{(j)}(1)} &\equiv \pm \frac{\bar{h}(\lambda_i^{(0)}) \prod_{j=1}^{e_i} h(\lambda_i^{(j)})}{2^{\lfloor \frac{t_i - t_i^{(0)} + 1}{2} \rfloor + \lfloor \frac{t_i^{(0)} - m_i^{(0)}}{2} \rfloor - 1}} \pmod{p} \\ &\equiv \pm \frac{1}{2^{\lfloor \frac{t_i - m_i^{(0)} + 1}{2} \rfloor - 1}} \bar{h}(Q_i) \pmod{p} \\ &\equiv \pm \frac{2}{2^{\lfloor \frac{t_i - m_i^{(0)}}{2} \rfloor}} \bar{h}(Q_i) \pmod{p} \end{aligned}$$

(since  $t_i - m_i^{(0)}$  is even). By similar arguments, we obtain, in all other cases,

$$\frac{|T_i^+|}{2^{\lfloor S_i/2 \rfloor} \prod_{j=0}^{e_i} \psi_i^{(j)}(1)} \equiv \pm \frac{2}{2^{\lfloor \frac{t_i - m_i^{(0)}}{2} \rfloor}} \bar{h}(Q_i) \pmod{p}.$$

Finally, we get

$$\frac{|N_G(X)|_{p'}}{\langle (Q_1, \dots, Q_k) \rangle(1)_{p'}} \equiv \pm \frac{2^k}{2^{k-1} 2^{\lfloor s/2 \rfloor}} \prod_{i=1}^k \frac{1}{2^{\lfloor (t_i - m_i^{(0)})/2 \rfloor}} \bar{h}(Q_i) \pmod{p},$$

as announced.  $\square$

**4.2. Isaacs-Navarro Conjecture.** We can now prove the main result of this section.

**Theorem 4.4.** *The Isaacs-Navarro Conjecture holds for the principal spin block of  $S^+(pw)$  via a sign-preserving bijection.*

*Proof.* Let  $B$  be the principal spin block of  $G = S^+(pw)$ , and  $b$  its Brauer correspondent in  $N_G(X)$ . Let  $pw = \sum_{i=1}^k t_i p^i$  be the  $p$ -adic decomposition of  $pw$ . By Proposition 2.7,  $\lambda \succ pw$  labels a spin character of  $B$  of  $p'$ -degree if and only if  $\lambda$  has  $\bar{p}$ -core tower  $(R_1^\lambda, \dots, R_k^\lambda)$  with  $|R_i^\lambda| = t_i$  for each  $1 \leq i \leq k$ . Also, for any such  $\lambda$ , we have, by Lemma 2.6,  $\sigma(\langle \lambda \rangle) = \sigma(\lambda) = \sigma(R_1^\lambda) \dots \sigma(R_k^\lambda)$ . By the above description of  $M(b)$ , this implies that

$$\Phi: \begin{cases} M(B) & \longrightarrow M(b) \\ \langle \lambda \rangle & \longmapsto \langle (R_1^\lambda, \dots, R_k^\lambda) \rangle \end{cases}$$

is a sign-preserving bijection. Furthermore, it is immediate from Proposition 2.8 and Proposition 4.3 that, for any  $\langle \lambda \rangle \in M(B)$ ,

$$\frac{|G|_{p'}}{\langle \lambda \rangle(1)_{p'}} \equiv \pm \frac{|N_G(X)|_{p'}}{\langle (Q_1, \dots, Q_k) \rangle(1)_{p'}} \pmod{p}.$$

This proves the result.  $\square$

## 5. MAIN THEOREM

We can now finally give our main theorem:

**Theorem 5.1.** *The Isaacs-Navarro Conjecture holds for all covering groups of the symmetric and alternating groups, whenever  $p$  is an odd prime.*

*Proof.* First, let  $G$  be any central extension of degree 2 of  $S(n)$  or  $A(n)$ , and  $B$  be a  $p$ -block of  $G$ . If  $B$  is an unfaithful block, then the Isaacs-Navarro Conjecture holds for  $B$  by the results of Fong ([2]) and Nath ([8]). If  $B$  is a spin-block of  $G$  of weight  $w > 0$ , then the Isaacs-Navarro Conjecture holds for  $B$  by Theorem 3.4 and Theorem 4.4. If  $w = 0$ , then  $B$  contains a unique spin character (of  $p$ -defect 0), and the result is immediate.

Finally, the case of the exceptional 6-fold covers of  $A(6)$  and  $A(7)$  can easily be checked using the character tables given in [6, 6. Appendix], or with a computer.  $\square$

**Acknowledgements.**

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